PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS
WITH TWO ASSOCIATE CLASSES AND THREE
REPLICATIONS

by

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Special report to the U. S. Air Force of work at Chapel Hill under Contract AF 18 (600) - 83 for research in probability and statistics.

Institute of Statistics
Mimeograph Series No. 54
For Limited Distribution

June, 1952
ACKNOWLEDGEMENT

I wish to express my sincere appreciation to Professor R. C. Bose for his enthusiastic encouragement and invaluable guidance of this research.

I wish also to thank the Institute of Statistics and the U. S. Air Force for financial assistance throughout the period of this investigation.

Willard H. Clatworthy
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>A NEW SERIES OF SYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1. Partially Balanced Incomplete Block Designs with Two Associate Classes</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2. Two Useful Lemmas</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>3. A New Series of Symmetrical Partially Balanced Designs</td>
<td>11</td>
</tr>
<tr>
<td>II</td>
<td>SYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES AND THREE REPLICATIONS</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>1. General Theory and Derivation of Parameters</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>2. The Impossibility of Design No. 15</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>3. Useful Lemmas for Partially Balanced Designs with Two Associate Classes and ( \lambda_1 = 1 ) and ( \lambda_2 = 0 )</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>4. The Impossibility of Designs No. 4, 5, 6, 7, 11, and 13</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>5. The Impossibility of Design No. 14</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>6. A Construction for Design No. 12</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>7. Summary</td>
<td>71</td>
</tr>
<tr>
<td>Chapter</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>-------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>III</td>
<td>DERIVATION OF THE PARAMETERS OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES AND $k &gt; r = 3$</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>1. Introduction</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>2. An Exhaustive Enumeration of Partially Balanced Incomplete Block Designs with Two Associate Classes and $k &gt; r = 3$</td>
<td>74</td>
</tr>
<tr>
<td>IV</td>
<td>UNSYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE CLASSES AND $k &gt; r = 3$</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>1. Introductory Remarks</td>
<td>90</td>
</tr>
<tr>
<td></td>
<td>2. The Relationship of Duality Between Members of a Certain Two-Parameter Series of Partially Balanced Designs</td>
<td>94</td>
</tr>
<tr>
<td></td>
<td>3. Constructions and Impossibility Proofs for Designs of Series IV and Their Duals</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>4. The Designs of Series IX</td>
<td>123</td>
</tr>
<tr>
<td></td>
<td>APPENDIX</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>135</td>
</tr>
</tbody>
</table>
INTRODUCTION

Experimentalists have recently shown considerable interest in designs having a relatively large number of treatments (or varieties) but a small number of replications (2, 3, 4, or 5).

The only incomplete block designs in which the difference between any two treatments is measured with equal accuracy are the balanced incomplete block designs. In these designs, the maximum number of treatments that can be accommodated when the number of replications is $r$ is $r^2 - r + 1$. It is therefore of interest to investigate incomplete block designs in which $r$ is small and two accuracies are involved in the estimation of treatment differences. The only general class of such designs for which a scheme of analysis is available is the class of partially balanced designs with two associate classes. Bose has completely exhausted the class of connected partially balanced designs with two associate classes and two replications.

The object of this investigation is to completely enumerate the arithmetically possible connected partially balanced incomplete block designs with two associate classes and three replications, and either to give constructions for the new designs having $3 \leq k \leq 10$ or else prove them impossible. This objective has been achieved.
It was hoped that this investigation would result in several new designs, thus making available to the experimenter a larger number of designs from which he could choose the most appropriate one for his particular experiment. As it turned out, there are only three new designs with \( r = 3 \) and all other designs were shown to be impossible. Several designs closely related to those under investigation were either solved or proved impossible. Thus, Chapter I contains a general solution of a series of symmetrical partially balanced designs having \( r = k = s + 1 \), where \( s \) is a prime or a positive integral power of a prime. Chapter IV contains an interesting series of designs having \( k = 3 \) and \( r \geq 2 \). These designs are the duals of designs having three replications.

For the convenience of the experimenter, an Appendix containing pertinent information concerning the new solvable designs having \( v < 100 \) and \( 3 \leq k \leq 10 \) has been added.
CHAPTER I

A NEW SERIES OF SYMMETRICAL PARTIALLY BALANCED
INCOMPLETE BLOCK DESIGNS

1. Partially Balanced Incomplete Block Designs with

Two Associate Classes

1.1. Partially balanced incomplete block designs with m
associate classes (m \geq 1) were introduced in 1939 by Bose and
Nair\(^1\). Balanced incomplete block designs and square
lattices were included as special cases. Later Nair and Rao
\(^{13}\) broadened the definition of partially balanced designs
so as to further include cubic and other higher dimensional
lattice designs. Bose and Shimamoto\(^8\) have recently re­
phrased the definition of a partially balanced design in order
to stress the fact that the relationships between the treatments
are determined only by the parameters \(n_1, n_2, \ldots, n_m\), and
\(p_{ijk}^1\) (i, j, k = 1, 2, ..., m). For designs with only two associate
classes (m = 2) the Bose and Shimamoto definition is substantially
as follows:

A partially balanced incomplete block design with two
associate classes is an arrangement of v treatments (or varieties)

\(^1\) Numbers in square brackets refer to bibliography.
in b blocks such that:

(i) Each of the v treatments is replicated r times in b blocks each of size k, and no treatment appears more than once in any block.

(ii) There exists a relationship of association between every pair of the v treatments satisfying the following conditions:

(a) Any two treatments are either first or second associates.

(b) Each treatment has n₁ first and n₂ second associates.

(c) Given any two treatments which are i-th associates, the number of treatments common to the j-th associates of the first and the k-th associates of the second is \( p_{jk}^i \) and this number is independent of the pair of treatments with which we start. Furthermore, \( p_{jk}^i = p_{kj}^i \) (i, j, k = 1, 2).

(iii) Any pair of treatments which are i-th associates occur together in exactly \( \lambda_i \) blocks (i = 1, 2).

In [13], the modification of the original definition [7] was to permit some of the parameters \( \lambda_1, \lambda_2, \ldots, \lambda_m \) to be equal. However, in the case of partially balanced designs with two associate classes, if \( \lambda_1 = \lambda_2 \) the design degenerates into a balanced incomplete block design, a class of incomplete block designs studied by many authors, including Yates [18], Fisher [9], and Bose [2]. Since the research of this writer is
concerned only with strictly partially balanced designs, we shall restrict ourselves to \( \lambda_1 \neq \lambda_2 \). If either \( n_1 \) or \( n_2 \) assumes the value zero, one associate class does not exist. Hence, we shall require that \( n_1 \) and \( n_2 \) be positive integers.

1.2. Bose and Nair \cite{BoseNair} have proved that the following relationships between the parameters of a partially balanced design with two associate classes are necessary:

\begin{align}
(1.1.21) & \quad v_r = b_k \\
(1.1.22) & \quad v = n_1 + n_2 + 1 \\
(1.1.23) & \quad \lambda_1 n_1 + \lambda_2 n_2 = r(k-1) \\
(1.1.24) & \quad p^1_{11} + p^1_{12} + 1 = p^2_{11} + p^2_{12} = n_1 \\
(1.1.25) & \quad p^1_{21} + p^1_{22} = p^2_{21} + p^2_{22} + 1 = n_2 \\
(1.1.26) & \quad n_1 p^2_{12} = n_2^2 p^2_{11}, \quad n_1 p^1_{22} = n_2^2 p^2_{12}.
\end{align}

They have also proved that if values are assigned to the parameters of the first kind \((v, b, r, k, \lambda_1, \lambda_2, n_1, \text{ and } n_2)\) satisfying \((1.1.21), (1.1.22), \text{ and } (1.1.23)\), then only one of the parameters of the second kind \((p^i_{jk}, i, j, k = 1, 2)\) is independent.

The parameters of the second kind will frequently be exhibited as elements of two symmetric matrices,
1.3. Recently Bose and Shimamoto classified into five distinct types all known partially balanced incomplete block designs with two associate classes. These five types or sub-classes are

1. Group divisible designs
2. Triangular designs
3. Singly-linked blocks designs
4. Latin square type with i constraints
5. Cyclic.

This classification is according to the 'association schemes' of the designs. The association schemes are explicit representations of the association relationships between the v treatments. In this dissertation new designs belonging to none of the above types are constructed. The association schemes of the five types of designs above have rather simple descriptions, whereas the new designs constructed by this writer have somewhat more complicated association schemes. The value of the association scheme lies in the fact that it simplifies the numerical computations in the analysis of the design as well as the interpretation of the results.
Since in this study reference needs to be made to the group divisible and triangular designs (especially the former), and since the papers by Bose and Shimamoto [8], and Bose and Connor [6] are at the time of writing still unpublished, the relevant definitions and properties are given here in brief.

A group divisible (GD) design is a partially balanced incomplete block design in which the v treatments can be divided into m groups of n treatments each such that two treatments belonging to the same group are first associates and two treatments belonging to different groups are second associates. The association scheme of a group divisible design consists of an m x n rectangular array in which each row contains only the n treatments of a group. Two treatments belonging to the same group are first associates while two treatments belonging to different groups are second associates. The following relationships hold:

$$n_1 = n - 1, \quad n_2 = n(m-1)$$

$$P_1 = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix}.$$ 

In [8] the group divisible designs are subdivided into three mutually exclusive and exhaustive types:
(1) Singular group divisible designs characterized by
\[ r = \lambda_1 \text{ and } rk = v\lambda_2 \geq 0, \]

(2) Semi-regular group divisible designs characterized
by \( r > \lambda_1 \text{ and } rk - v\lambda_2 = 0, \)

(3) Regular group divisible designs characterized
by \( r > \lambda_1 \text{ and } rk - v\lambda_2 > 0. \)

The three theorems stated below are proved in \([6,7]\).

**Theorem 1.** The necessary and sufficient condition for a
partially balanced design to be group divisible is the vanishing
of \( p_{12}^1 \) or \( p_{12}^2. \) If \( p_{12}^1 = 0 \) then the treatments in the same group
are \( i \)-th associates \((i = 1, 2).\)

**Theorem 2.** If in a balanced incomplete block design with
parameters \( v^*, b^*, r^*, k^*, \text{ and } \lambda^* \) each treatment is replaced by
a group of \( n \) treatments, we get a singular group divisible
design with parameters given by
\[ v = nv^*, \ b = b^*, \ r = r^*, \ k = nk^*, \ \lambda_1 = r^*, \ \lambda_2 = \lambda^*, \ m = v^*, \ n = n. \]
Conversely, every singular group divisible design is obtainable
in this way from a corresponding balanced incomplete block design.

**Corollary.** For a singular group divisible design \( b \geq m. \)

**Theorem 3.** For any group divisible design \( rk - v\lambda_2 > 0. \)

Bose, Bhattacharya, and Shrikhande \([5,7]\) have made an
extensive study of the construction of group divisible designs.
This paper is in preparation.
The triangular type of partially balanced incomplete block design is one having \( v = \frac{n(n-1)}{2} \) treatments and an association scheme in which the treatments can be arranged in an \( n \times n \) square with the cells along the main diagonal blocked out and the remaining cells filled so that the arrangement is symmetrical with respect to the main diagonal \( \sum \). Then to be a triangular design it is required that any pair of treatments lying in the same row or column of the \( n \times n \) square are first associates while treatments not lying in the same row or column are second associates. For triangular designs

\[
\begin{align*}
    n_1 &= 2n - 4, \\
    n_2 &= \frac{(n-2)(n-3)}{2}, \\
    p_1 &= \begin{pmatrix} n-2 & n-3 \\ n-3 & \frac{(n-3)(n-4)}{2} \end{pmatrix}, \\
    p_2 &= \begin{pmatrix} 4 & 2n-8 \\ 2n-8 & \frac{(n-4)(n-5)}{2} \end{pmatrix}.
\end{align*}
\]

2. Two Useful Lemmas

**Lemma 1.2.1.** Let there exist a relationship of association between every pair among the \( v \) treatments satisfying the conditions:

(a) Any two treatments are either first or second associates.

(b) Each treatment has \( n_1 \) first and \( n_2 \) second associates.
(c) For any pair of treatments which are first associates, the number of treatments common to the first associates of the first and the first associates of the second, \( p_{11} \), is independent of the pair of treatments with which we start.

Then, for every pair of first associates among the \( v \) treatments the numbers \( p_{12}, p_{21}, \) and \( p_{22} \) are constants, and \( p_{12} = p_{21} \).

**Proof:** Let \( \Theta \) and \( \phi \) be an arbitrarily chosen pair of first associates from among the \( v \) treatments. Let \( p^1_{jk}(\Theta, \phi) \) denote the number of treatments common to the \( j \)-th associates of \( \Theta \) and to the \( k \)-th associates of \( \phi \), \((j, k = 1, 2)\). The \( n_1 \) first associates of \( \Theta \) are made up of \( \phi \), the \( p^1_{11}(\Theta, \phi) \) treatments which are first associates of \( \Theta \) as well as \( \phi \), and the \( p^1_{12}(\Theta, \phi) \) treatments which are first associates of \( \Theta \) but second associates of \( \phi \). Hence,

\[
(1.2.1) \quad 1 + p^1_{11}(\Theta, \phi) + p^1_{12}(\Theta, \phi) = n_1.
\]

Likewise, classifying the first associates of \( \phi \), we have

\[
(1.2.2) \quad 1 + p^1_{11}(\Theta, \phi) + p^1_{21}(\Theta, \phi) = n_1.
\]

Similarly, the \( n_2 \) second associates of \( \Theta \) are made up of the \( p^1_{21}(\Theta, \phi) \) treatments which are second associates of \( \Theta \) and first associates of \( \phi \) and the \( p^1_{22}(\Theta, \phi) \) treatments which are second associates of both \( \Theta \) and \( \phi \). Hence,
But by hypothesis, $p_{11}(\Theta, \emptyset)$ is independent of the pair of treatments $\Theta$ and $\emptyset$ and is $p_{11}^1$. Hence, from (1.2.1), (1.2.2), and (1.2.3)

\begin{equation}
(1.2.4) \quad p_{12}^1(\Theta, \emptyset) = p_{21}^1(\Theta, \emptyset) = n_1 - p_{11}^1 - 1
\end{equation}

and

\begin{equation}
(1.2.5) \quad p_{22}^1(\Theta, \emptyset) = n_2 - n_1 + p_{11}^1 + 1.
\end{equation}

Since $\Theta$ and $\emptyset$ are an arbitrarily chosen pair of first associates, the relations (1.2.4) and (1.2.5) amount to a proof of the Lemma.

**Lemma 1.2.2.** Let there exist a relationship of association between every pair among the $v$ treatments satisfying the conditions:

(a) Any two treatments are either first or second associates.

(b) Each treatment has $n_1$ first and $n_2$ second associates.

(c) For any pair of treatments which are second associates, the number of treatments common to the first associates of the first and the first associates of the second, $p_{11}^2$, is
independent of the pair of treatments with which we start.

Then, for every pair of second associates among the $v$ treatments the numbers $p_{12}^2$, $p_{21}^2$, and $p_{22}^2$ are constants, and $p_{12}^2 = p_{21}^2$.

Proof is similar to that of Lemma 1.2.1. In fact it can be shown that

\[(1.2.6) \quad p_{12}^2(\emptyset, \emptyset) = p_{21}^2(\emptyset, \emptyset) = n_1 - p_{11}^2\]

and

\[(1.2.7) \quad p_{22}^2(\emptyset, \emptyset) = n_2 - n_1 + p_{11}^2 - 1,\]

where $\emptyset$ and $\emptyset$ are an arbitrarily chosen pair of second associates.

Insofar as partially balanced incomplete block designs with two associate classes are concerned, the consequence of Lemmas (1.2.1) and (1.2.2) is that the definition of a partially balanced design given by Bose and Shimamoto demands more than is needed. For partially balanced designs with two associate classes a less demanding definition could be formed by retaining (i), (a) and (b) of (ii), and (iii) of Subsection 1.1 and replacing condition (c) of (ii) by (c') given below.

(c') For any pair of the $v$ treatments which are $i$-th associates, the number of treatments common to the first associates of the first and the first associates of the second is $p_{11}^4(i = 1, 2)$,
and this number is independent of the pair of treatments with which we start.

This new definition and Lemmas 1.2.1 and 1.2.2 are then equivalent to the Bose and Shimamoto definition. Under the Bose and Shimamoto definition, if one wants to prove that an arrangement of objects is a partially balanced design, it is necessary insofar as (c) of (ii) is concerned to show the constancy of all eight parameters $p_{jk}^i (i, j, k = 1, 2)$ and also to show that the equalities $p_{jk}^i = p_{kj}^i (j \neq k; i,j,k = 1,2)$ hold. By use of Lemmas 1.2.1 and 1.2.2, with regard to the parameters of the second kind, it is only necessary to show the constancy of $p_{11}^1$ and $p_{11}^2$.

3. A New Series of Symmetrical Partially Balanced Designs

3.1. We shall now give a solution for the series of symmetrical partially balanced designs whose parameters are

\[
\begin{align*}
\lambda_1 &= 1, \quad \lambda_2 = 0, \quad n_1 = s^2 + s, \quad n_2 = s^3, \\
P_1 &= \begin{pmatrix} s-1 & s^2 \\ s^2(s-1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} s+1 & (s-1)(s+1) \\ s^2(s-1) \end{pmatrix}
\end{align*}
\]
where \( s = p^n \), \( p \) a prime and \( n \) a positive integer. It will be found that the association scheme for designs of this series does not belong to any of the five types given by Bose and Shimamoto, except for the case \( s = 2 \) which is known [8] and in which the association scheme is triangular.

Bose [2] and Bose and Nair [7] have used finite geometries and Galois fields in the construction of incomplete block designs. We shall obtain a general solution of the series of designs (1.3.11) by use of the finite projective geometry \( \text{PG}(3, s) \) in which the coordinates of points are elements of the Galois field \( GF(s) \).

The finite projective geometry \( \text{PG}(3, s) \) has \( (s^4-1)/(s-1) \) points, \( (s^2+1)(s^2+s+1) \) lines, and \( (s^4-1)/(s-1) \) planes. The number of points contained in a line is \( s+1 \) and the number of points in a plane is \( s^2+s+1 \). Through any point there pass \( s^2+s+1 \) lines and the same number of planes. One line and \( s+1 \) planes pass through every pair of points of \( \text{PG}(3, s) \). We identify the \( (s^4-1)/(s-1) \) points of \( \text{PG}(3, s) \) with the \( (s^4-1)/(s-1) \) treatments of design (1.3.11) such that a point of \( \text{PG}(3, s) \) corresponds to one and only one treatment of the design. To construct the design we establish a one-to-one correspondence between points and planes of \( \text{PG}(3, s) \) in the following manner. Let

\[
(1.3.12) \quad u_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4, \quad i = 1, 2, 3, 4,
\]
where $a_{ij}$ ($i, j = 1, 2, 3, 4$) belongs to the Galois field $GF(p^n)$ and the matrix $(a_{ij})$ is skew-symmetric, i.e., $a_{ii} = 0$ and $a_{ij} + a_{ji} = 0$, $i \neq j$. To the point $\Theta$ with coordinates $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ let there correspond the plane $\pi_\Theta$ whose equation is

\[(1.3.13) \quad \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0.\]

We shall now prove the following lemma:

**Lemma 1.3.11.** (i) The plane $\pi_\Theta$ passes through the point $\Theta$.

(ii) If the plane $\pi_\Theta$ passes through the point $\Theta$, then the plane $\pi_\phi$ passes through point $\Theta$.

**Proof (i):**

By (1.3.12) and (1.3.13), the equation of the plane $\pi_Q$ corresponding to point $Q = (1, 1, 1, 1, 1)$ can be expressed in the form

\[(1.3.14) \quad \sum_{i,j=1}^{4} a_{ij} \lambda_i x_j = 0.\]

Using the fact that the matrix $(a_{ij})$ is skew-symmetric, (1.3.14) becomes

\[(1.3.15) \quad a_{12}(\lambda_1 x_2 - \lambda_2 x_1) + a_{13}(\lambda_1 x_3 - \lambda_3 x_1) + a_{14}(\lambda_1 x_4 - \lambda_4 x_1) + a_{23}(\lambda_2 x_3 - \lambda_3 x_2) + a_{24}(\lambda_2 x_4 - \lambda_4 x_2) + a_{34}(\lambda_3 x_4 - \lambda_4 x_3) = 0.\]
Obviously, equation (1.3.15) is satisfied when

\[(1.3.16) \quad (x_1, x_2, x_3, x_4) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \emptyset.\]

Hence, point \emptyset lies on plane \(\pi_\emptyset\).

**Proof (ii):**

Let point \emptyset be denoted by \((m_1, m_2, m_3, m_4)\). Since plane \(\pi_\emptyset\) corresponding to point \emptyset passes through point \emptyset by hypothesis, we have from (1.3.14)

\[(1.3.17) \quad \sum_{i,j=1}^{h} a_{ij} m_i \lambda_j = 0.\]

The plane \(\pi_\emptyset\) corresponding to point \emptyset is

\[(1.3.18) \quad m_1 U_1 + m_2 U_2 + m_3 U_3 + m_4 U_4 = 0,\]

or

\[(1.3.19) \quad \sum_{i,j=1}^{h} a_{ij} m_i x_j = 0.\]

For plane \(\pi_\emptyset\) to pass through point \emptyset it is necessary that the coordinates of \emptyset satisfy (1.3.19), i.e.,

\[(1.3.110) \quad \sum_{i,j=1}^{h} a_{ij} \lambda_i \lambda_j = 0.\]

Since matrix \((a_{ij})\) is skew-symmetric, equation (1.3.110) becomes
(1.3.111) \[ \sum_{i,j=1}^{l} a_{ij} m_{ij} = 0, \]

which is identical to equation (1.3.17). Therefore, if plane \( \pi_{\theta} \) passes through point \( \emptyset \), then plane \( \pi_{\emptyset} \) passes through point \( \emptyset \).

3.2. We now establish an association scheme for design (1.3.11). We shall say that the point (or treatment) \( \emptyset \) is a first associate of point \( \Theta \) if the plane \( \pi_{\emptyset} \) corresponding to \( \Theta \) passes through point \( \emptyset \). Let \( \emptyset \) be a first associate of \( \Theta \). Then by Lemma 1.3.11, plane \( \pi_{\emptyset} \) passes through point \( \emptyset \); and hence, point \( \emptyset \) is by definition a first associate of point \( \emptyset \). Thus, the association relation is symmetrical as it should be. By definition, any point \( \alpha \) which is not a first associate of \( \Theta \) is its second associate.

We shall now prove that the condition (c) of paragraph (ii) of the definition of a partially balanced design is satisfied. (See Section 1.1.)

(i) Since there are \( s^2 + s \) points lying in \( \pi_{\emptyset} \) other than \( \emptyset \) itself, the number of first associates of \( \emptyset \) is

(1.3.21) \[ n_{1} = s^2 + s. \]
Since the total number of points in PU(3, s) is \( s^3 + s^2 + s + 1 \), the number of second associates of \( \Theta \) is

\[
(1.3.22) \quad n_2 = s^3.
\]

(ii) Let points \( \Theta \) and \( \Phi \) be first associates. Points which are first associates of both \( \Theta \) and \( \Phi \) must lie on the line of intersection of planes \( \pi_\Theta \) and \( \pi_\Phi \) passing through \( \Theta \) and \( \Phi \) respectively. This line passes through both \( \Theta \) and \( \Phi \). It contains \( s-1 \) points other than \( \Theta \) and \( \Phi \). Hence, the number of points common to the first associates of \( \Theta \) and \( \Phi \) is

\[
(1.3.23) \quad p_{11} = s - 1.
\]

Next, let \( \Theta \) and \( \Phi \) be second associates. The points which are first associates of both \( \Theta \) and \( \Phi \) must lie on the line of intersection of the planes \( \pi_\Theta \) and \( \pi_\Phi \) as before. However, this line no longer passes through points \( \Theta \) and \( \Phi \). Since a line contains \( s+1 \) points, the number of treatments common to the first associates of \( \Theta \) and \( \Phi \) is

\[
(1.3.24) \quad p_{11}^2 = s + 1.
\]

It now follows from Lemmas 1.2.1 and 1.2.2 and expressions (1.2.4), (1.2.5), (1.2.6), and (1.2.7) that
\[(1.3.25)\]
\[
p_{12} = p_{21} = n_1 - p_{11} - 1 = s^2,
\]

\[(1.3.26)\]
\[
p_{22} = n_2 - n_1 + p_{11} + 1 = s^2(s-1),
\]

\[(1.3.27)\]
\[
p_{12} = p_{21} = n_2 - p_{11} = (s-1)(s+1),
\]

and

\[(1.3.28)\]
\[
p_{22} = n_2 - n_1 + p_{11} - 1 = s^2(s-1).
\]

3.3. It remains to show that the conditions on parameters \(b, r, k, \lambda_1\) and \(\lambda_2\) of design \((1.3.11)\) are satisfied.

Let \(\ell_\theta\) be any line passing through \(\Theta\) and lying in the plane \(\pi_\theta\). The set of all points contained in line \(\ell_\theta\) is taken to form one block of our design. Since there are \(s+1\) lines passing through \(\Theta\) and lying in \(\pi_\theta\), there are \(s+1\) blocks containing treatment \(\Theta\). Hence,

\[(1.3.31)\]
\[
r = k = s + 1.
\]

The total number of points in \(P_0(3, s)\) is \((s^4-1)/(s-1)\). Hence, the total number of lines like \(\ell_\theta\) is \((s^4-1)/(s-1)\) since through each point there pass \(s+1\) lines like \(\ell_\theta\), and on each line there lie \(s+1\) points. Hence, the number of blocks is

\[(1.3.32)\]
\[
b = (s^4-1)/(s-1).
\]
If \( \Theta \) and \( \Phi \) are first associates, then the line joining \( \Theta \) and \( \Phi \) is a block. Hence, any two first associates occur together in exactly one block, giving

\[(1.3.33) \quad \lambda_1 = 1.\]

If \( \Theta \) and \( \Phi \) are second associates, the corresponding planes \( \pi_\Theta \) and \( \pi_\Phi \) are different. Hence, the lines \( \lambda_\Theta \) passing through \( \Theta \) never pass through \( \Phi \), and therefore, the blocks containing \( \Theta \) cannot contain \( \Phi \). Thus,

\[(1.3.34) \quad \lambda_2 = 0.\]

This completes the proof that the system of blocks defined above satisfies design (1.3.11).

3.4. We shall now construct the design of (1.3.11) corresponding to \( s = 3 \). This design has parameters

\[(1.4.21) \begin{cases} \nu = b = 40, \lambda_1 = 1, n_1 = 12, P_1 = \begin{pmatrix} 2 & 9 \\ 18 & \end{pmatrix}, \\ r = k = 4, \lambda_2 = 0, n_2 = 27, P_2 = \begin{pmatrix} 4 & 8 \\ 18 & \end{pmatrix}. \end{cases}\]

We use \( \text{PG}(3, 3) \) and the Galois field \( \text{GF}(3) \) whose \( s = 3 \) elements are 0, 1, and 2. Let the skew-symmetric matrix be
In order to have a one-to-one correspondence between sets of coordinates with elements in $\text{GF}(J)$ and the 40 points of $\text{PG}(J, J)$, we shall restrict ourselves to the set of coordinates in which the left-most non-zero digit is unity. In order to express the blocks of the design in terms familiar to the experimentalist, we set up a correspondence between the points of $\text{PG}(3, J)$ and the integers $1, 2, 3, \ldots, 40$ according to the following rule:

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ correspond to

$$\lambda_1 x_2 + 2\lambda_2 x_1 + 2\lambda_3 x_4 + \lambda_4 x_3 = 0.$$

In order to have a one-to-one correspondence between sets of coordinates with elements in $\text{GF}(3)$ and the 40 points of $\text{PG}(3, 3)$, we shall restrict ourselves to the set of coordinates in which the left-most non-zero digit is unity. In order to express the blocks of the design in terms familiar to the experimentalist, we set up a correspondence between the points of $\text{PG}(3, 3)$ and the integers $1, 2, 3, \ldots, 40$ according to the following rule:

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ correspond to

$$\lambda_1 s^3 + \lambda_2 s^2 + \lambda_3 s + \lambda_4 = \alpha$$

where the number $\alpha$ is defined by
\[ a = s^2 + s + 1 \text{ if } \lambda_1 = 1, \]
\[ a = s + 1 \quad \text{if } \lambda_1 = 0 \text{ and } \lambda_2 = 1, \]
\[ a = 1 \quad \text{if } \lambda_1 = \lambda_2 = 0 \text{ and } \lambda_3 = 1, \]
\[ a = 0 \quad \text{if } \lambda_1 = \lambda_2 = \lambda_3 = 0, \text{ and } \lambda_4 = 1. \]

The solution of design (1.4.21) is exhibited below in tabular form. It will be noted that only 13 of the 40 planes are exhibited in the solution. The points of the other 27 planes lie in the 13 planes indicated in column (2).
SOLUTION OF DESIGN (1.4.21)

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<td>15 13 29 37</td>
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CHAPTER II

SYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

WITH TWO ASSOCIATE CLASSES AND THREE REPLICATIONS

1. General Theory and Derivation of Parameters

1.1. A symmetrical partially balanced incomplete block design is one having \( v = b \) (or \( r = k \)). In this Chapter we shall develop some theory useful in deriving the parameters of designs (symmetrical or unsymmetrical). Since we are interested only in connected designs \( \leftarrow 1 \), we shall ignore any designs of the disconnected type. Furthermore, since we are interested in partially balanced designs which do not degenerate into balanced incomplete block designs, we shall consider only those cases for which \( \lambda_1 \neq \lambda_2 \). The convention, \( \lambda_1 > \lambda_2 \), is adopted.

1.2. Consider any partially balanced incomplete block design with two associate classes having

\[
\lambda_1 > \lambda_2 = 0.
\]

(2.1.21)

(1) Let us obtain a lower bound for the parameter \( p_{11} \).

Since \( \lambda_2 = 0 \), two second associates never occur together in a
block while any pair of treatments occurring in a block must be first associates. Consider any pair of first associates $\Theta$ and $\varnothing$. Treatments $\Theta$ and $\varnothing$ must occur together in at least one block. The other $k-2$ treatments appearing in any block with $\Theta$ and $\varnothing$ are first associates of both $\Theta$ and $\varnothing$. Hence,

$$(2.1.22) \quad k-2 \leq p_{11}^1.$$ 

(2) If in any partially balanced incomplete block design with two associate classes $p_{12}^1 = 0$, then the design must necessarily be of the group divisible type. (See Th. 1, Subsection 1.3, Chapter I.) Now it is easily seen that any group divisible design having

$$(2.1.23) \quad p_{12}^1 = 0, \quad \lambda_1 > \lambda_2 = 0$$

is such that the treatments belonging to any group never occur in blocks with treatments belonging to another group. Hence, the $b$ blocks divide up into disconnected sets of blocks. Since such designs do not possess the property of connectedness necessary for the estimation of all treatment contrasts, we need consider only those designs having

$$(2.1.24) \quad p_{12}^1 \geq 1.$$
From (1.1.24) and (2.1.22) it is clear that

$$(2.1.25) \quad p_{12}^1 \leq n_1 - k + 1.$$ 

Since $p_{12}^1 = p_{21}^1$ and $p_{22}^1$ is a non-negative integer, we obtain from (1.1.25)

$$(2.1.26) \quad p_{12}^1 \leq n_2.$$ 

Since (2.1.25) and (2.1.26) must hold simultaneously, we may write

$$(2.1.27) \quad 1 \leq p_{12}^1 \leq \min \{n_1 - k + 1, n_2\}.$$ 

(3) We now obtain bounds for the parameter $v$. From (1.1.26) and (1.1.22) we get

$$(2.1.28) \quad n_1^1 p_{12}^1 = (v - n_1 - 1)p_{11}^2.$$ 

Since $n_1, n_2, p_{12}^1 \geq 1$, it is seen from (2.1.28) that $p_{11}^2 \geq 1$. From (2.1.25) and (2.1.28) it is seen that

$$(2.1.29) \quad v \leq n_1^2 - n_1(k - 2) + 1.$$ 

Also from (1.1.22) we obtain the inequality

$$(2.1.210) \quad n_1 + 2 \leq v.$$ 

The results from (1), (2), and (3) are combined in the following theorem:
Theorem 2.1.21. For any connected partially balanced incomplete block design with two associate classes having \( \lambda_1 > \lambda_2 = 0 \)

(1) \( k - 2 \leq p_{11}^1 \),

(2) \( 1 \leq p_{12}^1 \leq \min \left\{ n_1 - k + 1, n_2 \right\} \),

and

(3) \( n_1 + 2 \leq v \leq n_1^2 - n_1(k - 2) + 1 \).

1.3. Next, consider any partially balanced incomplete block design with two associate classes having

(2.1.31) \( \lambda_1 > \lambda_2 > 0 \).

From (1.1.24) and (1.1.25) we see that the inequalities,

(2.1.32) \( p_{12}^1 \leq n_1 - 1 \)

and

(2.1.33) \( p_{12}^1 \leq n_2 \),

must hold simultaneously. Hence,

(2.1.34) \( 0 \leq p_{12}^1 \leq \min \left\{ n_1 - 1, n_2 \right\} \).
Solving (1.1.22) and (1.1.23) simultaneously, we obtain

\[(2.1.35) \quad n_1 = \frac{r(k - 1)}{\lambda_1 - \lambda_2} - \frac{\lambda_2(v - 1)}{\lambda_1 - \lambda_2} \]

and

\[(2.1.36) \quad n_2 = \frac{\lambda_1(v - 1)}{\lambda_1 - \lambda_2} - \frac{r(k - 1)}{\lambda_1 - \lambda_2} . \]

From (2.1.35) and (2.1.36) we obtain the following lower and upper bounds on \(v\):

\[(2.1.37) \quad \frac{r(k-1) + 2\lambda_1 - \lambda_2}{\lambda_1} \leq v \leq \frac{r(k-1) + 2\lambda_2 - \lambda_1}{\lambda_2} . \]

Hence, we may state:

**Theorem 2.1.31.** For any partially balanced incomplete block design with two associate classes having \(\lambda_1 > \lambda_2 > 0\)

\[(1) \quad 0 \leq p_{12} \leq \min \left\{ n_1 - 1, n_2 \right\} \]

and

\[(2) \quad \frac{r(k-1) + 2\lambda_1 - \lambda_2}{\lambda_1} \leq v \leq \frac{r(k-1) + 2\lambda_2 - \lambda_1}{\lambda_2} . \]
1.4. Consider any partially balanced incomplete block design with two associate classes having

\[(2.1.41) \quad r = \lambda_1 > \lambda_2.\]

The \(n_1\) first associates of any treatment \(Q\) of the design must occur in each of the \(r\) blocks containing \(Q\). The treatment \(Q\) and its \(n_1\) first associates constitute a group of treatments any one of which is a first associate of all the other treatments of the group. Hence, each block of the design consists of

\[(2.1.42) \quad k^* = k/n\]

groups where \(n = n_1 + 1\). Furthermore, two treatments belonging to different groups must be second associates (since if they are not second associates, then they are first associates and belong to the same group, a contradiction). Thus, the \(v\) treatments of the design may be partitioned into

\[(2.1.43) \quad m = v/n\]

groups such that any two treatments belonging to the same group are first associates while any two treatments belonging to different groups are second associates. But this is the definition of a group divisible design. Furthermore, the design must be of the singular type since \(r = \lambda_1\). Hence, we may state:
Theorem 2.1.41. Any partially balanced incomplete block design with two associate classes having

\[ r = \lambda_1 > \lambda_2 \]

is a singular group divisible design.

If, in addition, it is known that

\[ \lambda_2 = 0, \]

then no block of the design can contain treatments belonging to two or more groups and the design is disconnected. Thus, we have the following useful corollary to Theorem 2.1.41:

Corollary 2.1.41. Any partially balanced incomplete block design with two associate classes having

\[ r = \lambda_1 > \lambda_2 = 0 \]

is a disconnected design.

1.5. We shall now derive the parameters of all arithmetically possible connected symmetrical partially balanced incomplete block designs with two associate classes having three replications.

(1) Case \( \lambda_1 = 1, \lambda_2 = 0 \). By Theorem 2.1.21, designs belonging to the subclass characterized by \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \)
must satisfy the inequalities

\[(2.1.51) \quad \frac{1}{p_{11}} \geq 1, \quad 1 \leq p_{12} \leq \min \{4, n_2 \}, \quad 8 \leq v \leq 31;\]

and from (1.1.26) and (1.1.22) the relationship

\[(2.1.52) \quad 6p_{12} = (v - 7)p_{11}^2\]

holds. By successively assigning to \(v\) all possible values, and for each value of \(v\) assigning in turn all possible values of \(p_{12}^1\), we determine sets of parameter values involving \(v, p_{12}^1,\) and \(p_{11}^2\). By use of (1.1.24) and (1.1.25) the remaining parameters of the second kind are determined for each set mentioned above. From (1.1.25) it is clear that the inequality

\[(2.1.53) \quad 0 \leq p_{12}^2 = p_{21}^2 \leq n_2 - 1\]

must be satisfied. Discarding all sets of parameter values which fail to satisfy (2.1.53) as well as those for which \(p_{12}^2 = 0\) but for which the integral condition on the parameter \(m\) for group divisible designs is not satisfied, we obtain fourteen arithmetically possible designs. These are listed in Table I.

(2) Case \(\lambda_1 = 2, \lambda_2 = 0\). By Theorem 2.1.21, the parameters of designs belonging to the subclass having \(\lambda_1 = 2\) and \(\lambda_2 = 0\) must satisfy the inequalities

\[(2.1.54) \quad \frac{1}{p_{11}} \geq 1, \quad 1 \leq p_{12}^1 \leq \min \{1, n_2 - 7\}, \quad 5 \leq v \leq 7.\]
From (1.1.26), (1.1.22), and (2.1.54) the relationship

\[(v - 4)p_{11}^2 = 3\]  \hspace{1cm} (2.1.55)

holds. Proceeding in the manner described for Case (1), it is found that there is only one arithmetically possible design, namely, Design No. 15 of Table I.

(3) Case \(\lambda_1 = 3, \lambda_2 = 0\). By Corollary 2.1.41, this class contains only disconnected designs in which we have no interest.

(4) Case \(\lambda_1 = 2, \lambda_2 = 1\). By Theorem 2.1.31, we have the following conditions on the designs under consideration:

\[0 \leq p_{12} \leq \min \left\lfloor \frac{n_1 - 1}{n_2} \right\rfloor, \quad 5 \leq v \leq 6.\]  \hspace{1cm} (2.1.56)

From (1.1.22) and (1.1.23) we obtain

\[n_1 + n_2 = v - 1\]  \hspace{1cm} (2.1.57)

and

\[2n_1 + n_2 = 6.\]  \hspace{1cm} (2.1.58)

For the two possible values of \(v\) we obtain the corresponding values of \(n_1\) and \(n_2\) from (2.1.57) and (2.1.58). By use of (1.1.26) and (2.1.56) and the fact that \(p_{12}^2 \leq n_2 - 1\)
must be satisfied, it is seen that only two arithmetically possible
designs exist. They are Designs No. 16 and 17 of Table I.

(5) Case $\lambda_1 = 3, \lambda_2 = 1$. By Theorem 2.1.31, it is
seen that $v$ can assume only two possible values, 4 and 5. From
(1.1.22) and (1.1.23) it is seen that when $v = 4$ the integral
condition on $n_1$ is not satisfied. When $v = 5$, the only possible
value $\frac{1}{\lambda}$ can assume is zero, which leads to failure to satisfy
the integral condition on the parameter $m$ for group divisible
designs. Hence, the subclass of symmetrical partially balanced
incomplete block designs with two associate classes and having
$r = k = 3, \lambda_1 = 3, \lambda_2 = 1$ is empty.

(6) Case $\lambda_1 = 3, \lambda_2 = 2$. The subclass of symmetrical
partially balanced incomplete block designs with two associate
classes and having $r = k = 3, \lambda_1 = 3, \lambda_2 = 2$ is also empty,
since by Theorem 2.1.31, we have

(2.1.59) $4 \leq v \leq 3$,

an absurdity.

We have now completely exhausted the class of connected
symmetrical partially balanced incomplete block designs with
two associate classes and three replications. The seventeen
arithmetically possible designs are given in Table I.
### TABLE I

**SYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS**

**WITH TWO ASSOCIATE CLASSES AND THREE REPLICATIONS**

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<th>Ref. No.</th>
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<td>8</td>
<td>13</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>Solution known</td>
</tr>
<tr>
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<td>15</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>8</td>
<td>1</td>
<td>3</td>
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</tr>
<tr>
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<td>3</td>
<td>1</td>
<td>0</td>
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<td>2</td>
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<td>Solution known</td>
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<tr>
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<td>19</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>3</td>
<td>1</td>
<td>New, impossible</td>
</tr>
<tr>
<td>12</td>
<td>19</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>1</td>
<td>2</td>
<td>New, solved</td>
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<td>1</td>
<td>0</td>
<td>6</td>
<td>18</td>
<td>2</td>
<td>1</td>
<td>New, impossible</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
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<td>24</td>
<td>1</td>
<td>1</td>
<td>New, impossible</td>
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<td>3</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>New, impossible</td>
</tr>
<tr>
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<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>Trivial</td>
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<tr>
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<td>6</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>Solution known</td>
</tr>
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</table>
1.6. Designs No. 1, 2, 3, and 10 of Table I were solved by Bose and Nair [7]; No. 8 was solved by Shrikhande [16]; and No. 9, which is the geometrical design with $s = 2$ in Chapter I, was solved by Bose and Shimamoto [8]. Design No. 16 is trivial. Designs No. 1, 2, and 17, which are of the group divisible type, have been treated in [5]. Additional information concerning the association schemes of Designs No. 3, 8, and 10 may be found in [15].

A proof of the impossibility of Design No. 15 is given in Section 2. Section 4 contains impossibility proofs for Designs No. 4, 5, 6, 7, 11, and 13, and Section 5 contains an impossibility proof for Design No. 14. A construction for Design No. 12 is given in Section 6.

2. The Impossibility of Design No. 15

Design No. 15 has parameters

\[
\begin{align*}
\nu &= b = 7, \quad \lambda_1 = 2, \quad n_1 = 3, \quad p_1 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \\
\lambda_2 &= 0, \quad n_2 = 3, \quad p_2 = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

(2.2.1)

Let $\Theta$ be any treatment of the design; let the three first associates of $\Theta$ be represented by the letters $a$, $b$, and $c$; and
let the three second associates of $\Theta$ be denoted by the integers 1, 2, and 3. Since $\lambda_1 = 2$, each of the pairs $(\Theta, a)$, $(\Theta, b)$, and $(\Theta, c)$ must occur in two blocks. Since $r = 3$, $\Theta$ must occur in three blocks; and since $\lambda_2 = 0$, none of these blocks can contain treatments 1, 2, or 3. Therefore, the two blocks containing the pair $(\Theta, a)$ must each contain either $b$ or $c$. One of these blocks must contain treatment $b$ and the other must contain treatment $c$, since if either treatment occurs in both blocks containing pair $(\Theta, a)$, then the third block containing treatment $\Theta$ would of necessity contain the same lettered treatment twice in contradiction to the definition of a partially balanced incomplete block design. Hence, the remaining block containing treatment $\Theta$ must also contain treatment $b$ and $c$, giving blocks

$$
\begin{array}{ccc}
\Theta & a & b \\
\Theta & a & c \\
\Theta & b & c
\end{array}
$$

Since $\lambda_2 = 0$, any two treatments which occur together in a block must be first associates. Hence, the three first associates of treatment $a$ are $\Theta$, $b$, and $c$. Since $\Theta$ and $a$ are first associates of each other, and since treatments $b$ and $c$ are common to the first associates of both $\Theta$ and $a$, $p_{11}^1 \geq 2$. But this contradicts the hypothesis that $p_{11}^1 = 1$. Hence, the design is combinatorially impossible.
3. Useful Lemmas for Partially Balanced Designs with Two Associate Classes and $\lambda_1 = 1$ and $\lambda_2 = 0$.

First, we shall introduce the notation and terminology which will be used in the remainder of this Chapter and those parts of Chapter IV where we are dealing with partially balanced designs with two associate classes characterized by the conditions $\lambda_1 = 1$ and $\lambda_2 = 0$.

Let the Greek letter $\Theta$ represent any treatment of the design. The $n_1 = r(k-1)$ first associates of $\Theta$ will be denoted by $r$ Latin letters each bearing a subscript $1, 2, \ldots, k-1$. The $k-1$ first associates of $\Theta$ appearing together with $\Theta$ in a block will be denoted by the same letter, and first associates of $\Theta$ appearing in different blocks with $\Theta$ will be denoted by different letters. The $n_2$ second associates of $\Theta$ will be denoted by the integers $1, 2, \ldots, n_2$. The $n_1$ first associates of $\Theta$ will sometimes be referred to as lettered treatments and the $n_2$ second associates of $\Theta$ as numbered treatments. The $r$ blocks containing treatment $\Theta$ will be referred to as the $\Theta$-blocks. Likewise, the blocks containing the lettered treatments $x_j$ ($x = a$ or $b$ or $c$, etc. and $j = 1, 2, \ldots, k-1$) but not $\Theta$ will be referred to as the $x$-blocks, and the blocks containing the numbered treatment $i$ ($1 \leq i \leq n_2$) will be called the $i$-blocks. Blocks containing only first associates of $\Theta$ or only second associates of $\Theta$ will be called
pure blocks while blocks containing both first and second associates of $q$ will be called mixed blocks.

In the special case of designs having $\lambda_1 = 1$, $\lambda_2 = 0$, and $p_{11}^l = k-2$, it will be shown that there are exactly $r(r-1)(k-l)$ mixed blocks each of which contains one lettered treatment and $k-l$ numbered treatments. Those $(r-1)(k-l)$ mixed blocks containing first associates of $q$ denoted by the same letter (and subscripts 1, 2, ..., $k-l$) will be called a group of blocks. There are $r$ such groups of blocks. The group of blocks containing the lettered treatment $x$ with any subscript will be referred to as the $x$-group of blocks ($x = a, b, c, etc.$). Within the $x$-group of blocks there are $r-1$ blocks containing the lettered treatment $x_j$. These $r-1$ blocks will be referred to as the $x_j$-set of blocks.

The method of attack in obtaining an impossibility proof is to assume the existence of the design and proceed to show that this leads to a contradiction. The method is so executed that if no contradiction arises one is led to a construction of the design.

We shall now prove some lemmas and corollaries which will be useful in the remainder of this Chapter as well as in Chapter IV.

**Lemma 2.3.1.** For a partially balanced incomplete block design with two associate classes having $\lambda_1 = 1$, $\lambda_2 = 0$, and $p_{22}^2 = 1$, the parameter $n_2$ must be even and
1) \( n_1 \geq 2(n_2-2) \)

and

2) \( p_{l1}^2 \geq n_2-2 \).

**Proof:** Let us examine the first and second associates of any numbered treatment \( i \) (1 \( \leq i \leq n_2 \)). Since \( \theta \) and \( i \) are second associates of each other and since \( p_{22}^2 = 1 \), the \( n_2 \) second associates of \( i \) must consist of one numbered treatment \( j \) (\( j \neq i \)), \( \theta \), and \( n_2-2 \) lettered treatments. The \( n_2 \) second associates of \( j \) include \( i, \theta \), and \( n_2-2 \) lettered treatments all of which are different from the lettered second associates of \( \theta \). Hence,

\[ n_1 \geq 2(n_2-2). \]

Furthermore, the \( n_2-2 \) other numbered treatments (excepting \( i \) and \( j \)) must be first associates of \( i \) as well as of \( j \). Since \( i \) and \( j \) are second associates of each other,

\[ p_{l1}^2 \geq n_2-2. \]

Now, \( i \) represents an arbitrarily chosen second associate of \( \theta \). Consequently, the \( n_2 \) second associates of \( \theta \) (numbered treatments) must divide up into pairs such that two numbered treatments within a pair are second associates of each other and two numbered treatments belonging to different pairs are first associates of each other. Hence, \( n_2 \) must be an even integer.
Lemma 2.3.2. For a partially balanced incomplete block design with two associate class having \( \lambda_1 = 1, \lambda_2 = 0, p_{11}^1 = k-1 \), and \( k > 2 \), the parameter \( p_{11}^2 \) must satisfy the condition \( p_{11}^2 \geq 2 \).

Proof: Let the \( r \) blocks containing treatment \( \Theta \) be

\[
\begin{align*}
\Theta & a_1 \quad a_2 \cdots a_{k-1} \\
\Theta & b_1 \quad b_2 \cdots b_{k-1} \\
& \vdots \quad \vdots \cdots \vdots \\
\Theta & \ell_1 \quad \ell_2 \cdots \ell_{k-1}
\end{align*}
\]

where the lettered treatments \( x_j \) (\( x = a, b, \ldots, \ell \); \( j = 1, 2, \ldots, k-1 \)) are the \( n_1 = r(k-1) \) first associates of \( \Theta \).

Consider any first associate \( x_j \) of \( \Theta \). Since \( r = 3, p_{11}^1 = k-1 \), and \( \Theta \) and \( x_j \) are first associates of each other, \( x_j \) must appear in just two other blocks, one of which must contain another first associate of \( \Theta \) which we shall call \( y_1 \) (\( x \neq y, 1 \leq i \leq k-1 \)). Since \( k > 2 \), this block must contain at least one numbered treatment, say \( t \) (\( 1 \leq t \leq n_2 \)). Now \( t \) and \( \Theta \) are second associates of each other, and \( x_j \) and \( y_1 \) are first associates of both of them.

Hence,

\[ p_{11}^2 \geq 2. \]
Lemma 2.3.3. For a partially balanced incomplete block design with two associate classes having $\lambda_1 = 1$, $\lambda_2 = 0$, and $p_{11}^1 = k-2$, any two treatments appearing in different blocks containing a common treatment must be second associates of each other.

Proof: There is no loss in generality if the treatment common to the two blocks is taken as treatment $\Theta$. Let $x_i$ and $y_j$ ($x \neq y; i, j = 1, 2, ..., k-1$) be any two treatments appearing in different blocks containing $\Theta$. Since $\lambda_1 = 1$ and $\lambda_2 = 0$, two treatments appearing in the same block are first associates of each other while any pair of treatments which do not occur together in any block are second associates of each other. The $k-2$ other treatments appearing in the same block with the pair $(\Theta, x_i)$ are first associates of both $\Theta$ and $x_i$. Since $p_{11}^1 = k-2$, all other first associates of $\Theta$ (including $y_j$) must be second associates of $x_i$. This proves the lemma.

Corollary 2.3.1. Under the conditions of Lemma 2.3.3 each mixed block contains only one lettered treatment, and the $r(r-1)(k-1)$ mixed blocks can be divided into $r$ distinct groups, each group containing $k-1$ sets and each set containing $r-1$ blocks.
Proof: Let the \( r \) \( \Theta \)-blocks of the design be

\[
\begin{align*}
\Theta & \ a_1 \ a_2 \ldots \ a_{k-1} \\
\Theta & \ b_1 \ b_2 \ldots \ b_{k-1} \\
\ldots & \ \ldots \ \ldots \\
\Theta & \ l_1 \ l_2 \ldots \ l_{k-1} \\
\end{align*}
\]

Since \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \), no pair of treatments can occur together in more than one block. By Lemma 2.3.3 any pair of lettered treatments \( x_i \) and \( y_j \) (\( x \neq y \)) must be second associates of each other. Hence, no two lettered treatments can appear together in a block free from \( \Theta \). Therefore, the design must contain \( r(r-1)(k-1) \) mixed blocks each of which contains one lettered and \( k-1 \) numbered treatments. Since each lettered treatment must appear in \( r-1 \) blocks free from \( \Theta \), there are \( (r-1)(k-1) \) mixed blocks containing treatments \( x_1, x_2, \ldots, x_{k-1} \). These blocks constitute the \( x \)-group of blocks (\( x = a, b, \ldots, l \)), and there are \( r \) distinct groups of blocks. Within the \( x \)-group there are \( r-1 \) blocks containing the lettered treatment \( x_1 \), and these we have called the \( x_1 \)-set of blocks (\( i = 1, 2, \ldots, k-1 \)). Each group of blocks obviously contains \( k-1 \) distinct sets of blocks.

**Corollary 2.3.2.** Under the conditions of Lemma 2.3.3 the numbered treatments appearing in a group of blocks must all be distinct. Hence,
If 

\[ n_2 \geq (r-1)(k-1)^2. \]

each group must contain precisely one complete replication of the numbered treatments.

Proof: Since \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \), it is clear that no numbered treatment can appear in two blocks belonging to the same set. Suppose the same numbered treatment \( j \) (\( j = 1, 2, \ldots, n_2 \)) appears in blocks belonging to different sets of the same group. Let the lettered treatment in one of the sets containing \( j \) be \( x_m \), and the lettered treatment in the other set containing \( j \) be \( x_n \) \((m \neq n; m, n = 1, 2, \ldots, k-1)\). Then \( x_m \) and \( x_n \) are first associates of each other since they appear together in the same \( \Theta \)-block. By Lemma 2.3.3, \( x_n \) and \( x_m \) must be second associates of each other since they appear in different blocks containing the common treatment \( j \), which contradicts the preceding sentence. Therefore, no numbered treatment can appear in two or more blocks belonging to the same group. Hence, the numbered treatments appearing in the same group of blocks must all be distinct. Since each mixed block contains \( k-1 \) numbered treatments, \( (r-1)(k-1)^2 \) distinct numbered treatments appear in a group of blocks. Since numbered treatments are second associates of \( \Theta \),

\[ n_2 \geq (r-1)(k-1)^2. \]
If \( n_2 = (r-1)(k-1)^2 \), each group must obviously contain exactly one complete replication of the numbered treatments.

**Corollary 2.3.3.** Under the conditions of Lemma 2.3.3 the mixed blocks contain \( p_{11}^2 \) complete replications of the numbered treatments. The pure blocks must contain \( r-p_{11}^2 \) complete replications of the numbered treatments.

**Proof:** Consider treatment \( \emptyset \) and an arbitrarily chosen numbered treatment \( i \) \( (i = 1, 2, \ldots, n_2) \). Since \( \emptyset \) and \( i \) are second associates of each other, and since \( \emptyset \) has only lettered treatments as first associates, treatment \( i \) must have exactly \( p_{11}^2 \) lettered first associates. Since lettered treatments appear only in \( \Theta \)-blocks and mixed blocks, and since no numbered treatment can appear in a \( \Theta \)-block, it follows by Corollary 2.3.1 that treatment \( i \) must appear in exactly \( p_{11}^2 \) mixed blocks. This is true for \( i = 1, 2, \ldots, n_2 \); therefore, the mixed blocks contain exactly \( p_{11}^2 \) complete replications of the numbered treatments.

Now, there are, in all, \( r \) complete replications of the numbered treatments. Therefore, the pure blocks consist of \( r-p_{11}^2 \) complete replications of the numbered treatments.

**Corollary 2.3.4.** Under the conditions of Lemma 2.3.3, two sets of blocks belonging to different groups must intersect in \( p_{11}^2 - 1 \) numbered treatments, i.e., must have \( p_{11}^2 - 1 \) numbered treatments in common.
Proof: By Lemma 2.3.3 and Corollary 2.3.1 the pair of lettered treatments $x_i$ and $y_j$ ($x \neq y$) appearing in two sets belonging to different groups are second associates of each other. Since $x_i$ and $y_j$ have 0 common to their first associates but no common lettered first associates, they must have $p^2_{11} - 1$ common numbered first associates. Since $x_i$ and $y_j$ appear only in the 0 blocks and in the sets under consideration, these two sets must contain the $p^2_{11} - 1$ common numbered first associates of $x_i$ and $y_j$. This proves the corollary.

Corollary 2.3.5. Under the conditions of Lemma 2.3.3 a mixed block cannot intersect a set belonging to another group in more than one treatment. If

$$n_2 = (r-1)(k-1)^2,$$

then each mixed block must intersect each set of another group in exactly one numbered treatment.

Proof: Suppose a mixed block intersects a set of another group in two or more treatments. Then consider a pair of numbered treatments common to the given mixed block and the intersected set of blocks belonging to another group. These two numbered treatments are first associates of each other since they occur together in a block. Their common first associates will include the $k-2$ other treatments in the block containing them both and
also the lettered treatment occurring in all blocks of the set intersected. Hence, \( p^1_{ll} \geq k-1 \) in contradiction to the hypothesis. This proves the first part of the corollary.

Since each group must contain a complete replication of numbered treatments when \( n_2 = (r-1)(k-1)^2 \), and since we have proved that a mixed block cannot intersect a set of another group in more than one treatment, it follows that when \( n_2 = (r-1)(k-1)^2 \) the \( k-1 \) numbered treatments in a mixed block must be distributed one to each set in the other groups. This proves the second part of the corollary.

4. The Impossibility of Designs No. 4, 5, 6, 7, 11, and 13

4.1. Design No. 4 has parameters

\[
\begin{align*}
\{ & v = b = 11, \lambda_1 = 1, n_1 = 6, p_1 = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 4, p_2 = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

Let the \( \Theta \)-blocks be

\[
\begin{align*}
\Theta & a_1 a_2 \\
\Theta & b_1 b_2 \\
\Theta & c_1 c_2
\end{align*}
\]
Since \( p_{11}^1 = 3 \), \( a_i \) \((i = 1, 2)\) has exactly two first associates from among \( b_1, b_2, c_1 \), and \( c_2 \). A similar result holds for \( b_i \) and \( c_i \) \((i = 1, 2)\). Hence, except for pairs occurring in the \( R \)-blocks, the lettered treatments form exactly six pairs. This can happen in three different ways, I, II(A), and II(B), and we shall show that in each case it is impossible to place the numbered treatments \((1, 2, 3, \text{and } 4)\) so as to satisfy the conditions of the design.

I. Suppose there are no pure blocks containing only lettered treatments. Then each of the six pairs still to be formed by the lettered treatments must occur in one of the remaining blocks. This accounts for all the lettered treatments, and leaves two blocks which must contain only numbered treatments. This is impossible unless a pair of numbered treatments is repeated in which case the conditions \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \) provide a contradiction.

II. Suppose there is at least one pure block containing lettered treatments. Without loss of generality we can take this block to be

\[
\begin{array}{ccc}
a_1 & b_1 & c_1 \\
\end{array}
\]

Since this block provides the necessary lettered pairs involving treatments \( a_1, b_1, \text{and } c_1 \), there must be three mixed blocks
where the asterisks represent numbered treatments. The treatments \(a_2, b_2,\) and \(c_2\) must form three pairs. Hence, the remaining four blocks can occur in only two ways:

(A) | (B)
--- | ---
\(a_2\) \(b_2\) \(c_2\) | \(a_2\) \(b_2\) *
\(a_2\) * * | \(a_2\) \(c_2\) *
\(b_2\) * * | \(b_2\) \(c_2\) *
\(c_2\) * * | * * *

where, as before, the asterisks represent numbered treatments. Now, \(a_1\) and \(a_2\) are first associates and must have three common first associates. Since \(\Theta\) is a common first associate of both \(a_1\) and \(a_2\) and since they have no common lettered first associates, they must have two common numbered first associates, say 1 and 2.

In case (A), treatments 1 and 2 must both occur in blocks \((a_1 * *)\) and \((a_2 * *)\), so that the pair \((1, 2)\) is repeated, contradicting the conditions \(\lambda_1 = 1\) and \(\lambda_2 = 0\).

In case (B), treatments 1 and 2 must both occur in the block \((a_1 * *)\), and the block \((a_2 b_2 *)\) must contain either 1 or 2 while the block \((a_2 c_2 *)\) contains the other. However,
a_1 and b_2 which are second associates have four common first associates, namely, treatments \( \Theta, a_2, b_1 \) and either 1 or 2. Since \( p_{11}^2 = 3 \), the design is impossible.

4.2. Design No. 5 has parameters

\[
\begin{align*}
v &= b = 13, \quad \lambda_1 = 1, \quad n_1 = 6, \quad p_1 = \begin{pmatrix} 4 & 1 \\ 5 & \end{pmatrix}, \\
r &= k = 3, \quad \lambda_2 = 0, \quad n_2 = 6, \quad p_2 = \begin{pmatrix} 1 & 5 \\ 0 & \end{pmatrix}.
\end{align*}
\]

We may write immediately the three \( \Theta \)-blocks containing the six first associates of \( \Theta \):

\[
\begin{align*}
\Theta & \quad a_1 \quad a_2 \\
\Theta & \quad b_1 \quad b_2 \\
\Theta & \quad c_1 \quad c_2.
\end{align*}
\]

Consider any lettered treatment \( x_i \) (\( x = a, b, c; i = 1, 2 \)) and its relationship with \( \Theta \). Since \( r = 3 \) and \( p_{11}^1 = 4 \), there must be two more blocks containing \( x_i \), and these two blocks must contain three other lettered treatments. Hence, one of these blocks must be a pure block of lettered treatments while the other must be a mixed block containing two lettered and one numbered treatment. Then, the numbered treatment must have at least two lettered treatments as first associates. Therefore, \( p_{11}^2 \geq 2 \) in contradiction to the hypothesis. Hence, Design No. 5 is combinatorially impossible.
4.3. The impossibility of Design No. 6, having parameters

\[
\begin{align*}
\{ & v = b = 13, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 1 & 4 \\ 2 & \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 6, P_2 = \begin{pmatrix} 4 & 2 \\ 3 & \end{pmatrix}, \\
\end{align*}
\]

follows immediately from Corollary 2.3.2 which requires that \( n_2 \geq 8 \).

4.4. The parameters of Design No. 7 are

\[
\begin{align*}
\{ & v = b = 13, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 3 & 2 \\ 4 & \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 6, P_2 = \begin{pmatrix} 2 & 4 \\ 1 & \end{pmatrix}. \\
\end{align*}
\]

Since \( \lambda_1 = 1, \lambda_2 = 0, \) and \( p_{22}^2 = 1 \), the conditions of Lemma 2.3.1 are satisfied. Hence, the existence of a solution requires that \( n_1 \geq 8 \) and \( p_{11}^2 \geq 4 \). Since Design No. 7 does not satisfy these requirements, it is combinatorially impossible.

4.5. Design No. 11 has parameters

\[
\begin{align*}
\{ & v = b = 19, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 3 & 2 \\ 10 & \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 12, P_2 = \begin{pmatrix} 1 & 5 \\ 6 & \end{pmatrix}. \\
\end{align*}
\]
We write immediately the three \( Q \)-blocks:

\[
\begin{array}{c}
Q \quad a_1 \quad a_2 \\
Q \quad b_1 \quad b_2 \\
Q \quad c_1 \quad c_2
\end{array}
\]

Since \( p^2_{l1} = 1 \), each numbered treatment must be a first associate of one and only one lettered treatment. Consequently, each mixed block must contain one lettered and two numbered treatments. There are two remaining blocks containing \( a_1 \), and since \( p^1_{l1} = 3 \), these blocks must contain just two lettered treatments other than \( a_2 \). Hence, they must contain two numbered treatments which must necessarily appear in the same block. Thus, the two blocks containing \( a_1 \) can be written as

\[
\begin{array}{c}
a_1 \quad * \quad * \\
a_1 \quad x \quad y
\end{array}
\]

where the asterisks represent numbered treatments and \( x \) and \( y \) are lettered treatments. A similar argument shows that the two remaining blocks containing \( a_2 \) are

\[
\begin{array}{c}
a_2 \quad * \quad *\\
a_2 \quad - \quad -
\end{array}
\]

where the asterisks represent numbered treatments different from
those in the mixed block \((a_1 \ast \ast)\) and where the blanks represent lettered treatments. Since \(p_{11}^1 = 3\) and \(a_1\) and \(a_2\) are first associates, the lettered treatments represented by the blanks can be none other than \(x\) and \(y\). This contradicts \(\lambda_1 = 1\) and \(\lambda_2 = 0\) and thus proves the impossibility of the design.

4.6. The impossibility of Design No. 13, having parameters

\[
\begin{align*}
\{ & v = b = 25, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 2 \\ 15 \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 18, P_2 = \begin{pmatrix} 1 \\ 12 \end{pmatrix}, \}
\end{align*}
\]

(2.4.61)

follows immediately from Lemma 2.3.2 which requires that \(p_{11}^2 \geq 2\).

5. The Impossibility of Design No. 14

For the impossibility proof of Design No. 14, having parameters

\[
\begin{align*}
\{ & v = b = 31, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 1 \\ 20 \end{pmatrix}, \\
& r = k = 3, \lambda_2 = 0, n_2 = 24, P_2 = \begin{pmatrix} 1 \\ 18 \end{pmatrix}, \}
\end{align*}
\]

(2.5.1)

we shall need two special lemmas which we now proceed to establish.

Since \(\lambda_1 = 1\), \(\lambda_2 = 0\), and \(p_{11}^1 = k-2\), Lemma 2.3.3 and Corollaries 2.3.1 through 2.3.5 apply. Hence, the 31 blocks of
the design consist of the three Q-blocks, twelve mixed blocks each of which contains exactly one first associate of Q (lettered treatment) and two second associates of Q (numbered treatments) and which altogether contain precisely one complete replication of the numbered treatments, and sixteen pure blocks which contain exactly two complete replications of the numbered treatments.

Lemma 2.5.1. For Design No. 14 to exist, it is necessary that each pure block of numbered treatments intersect each group of mixed blocks in exactly one treatment.

Proof: Suppose a pure block (i, j, k) consisting of three different numbered treatments i, j, and k intersects a group of mixed blocks in two treatments. Obviously, these two treatments must occur in different blocks of the group. Let the two intersected blocks of the group be

\[ \begin{align*}
  x & \quad i \quad * \\
  y & \quad j \quad *
\end{align*} \]

where x and y represent lettered treatments and the asterisks represent numbered treatments in which we have no special interest. Write the first associates of x, y, i and j arising from the three blocks in question. Since the lettered treatments belonging to the same group occur in the same Q-blocks, treatments x and y are either first associates or else they are identical.
If $x \neq y$, then $y$ and $i$ are first associates of both $x$ and $j$, violating condition $p_{21}^2 = 1$. If $x = y$, then $x$ and $k$ are first associates of $i$ and $j$, violating condition $p_{11}^1 = 1$. Hence, no pure block of numbered treatments can intersect any group in two treatments. Since there are just three groups and $i$, $j$, and $k$ each appear just once in the mixed blocks, it follows that the treatments $i$, $j$, and $k$ must be distributed one to each group. Hence, if Design No. 14 is to exist, it is necessary that each pure block of numbered treatments intersect each group of mixed blocks in exactly one treatment.

**Lemma 2.5.2.** If, under the conditions of Design No. 14, $(x \quad i_1 \quad i_2)$ and $(y \quad j_1 \quad j_2)$ are mixed blocks of different groups where $x$ and $y$ are lettered treatments and $i_1$, $i_2$, $j_1$, and $j_2$ are numbered treatments, then one and only one of the four pairs

$(2.5.2) \quad (i_1, j_1), (i_1, j_2), (i_2, j_1), \text{ and } (i_2, j_2)$

must occur among the pure blocks.
Proof: If possible, let the pairs \((i_1, j_1)\) and \((i_1, j_2)\) occur among the pure blocks. Then \(j_1\) and \(j_2\) which are first associates of each other (since they both occur in the block containing \(y\)) have the common first associates \(y\) and \(i_1\), contradicting the condition \(p_{il}^{1} = 1\). In the same way it can be shown that no two of the four pairs in (2.5.2) can occur among the pure blocks. Now, from Lemma 2.5.1, a pair of numbered treatments occurring together in a pure block cannot both occur in the same group of mixed blocks. Also, a pair of blocks belonging to different groups can be selected in exactly \(48\) ways which is the number of pairs in the 16 pure blocks. This shows that just one of the four pairs (2.5.2) occurs among the pure blocks.

Without loss of generality we may write the three \(\emptyset\)-blocks and the twelve mixed blocks as shown below.

\[
\begin{array}{ccc}
\emptyset & a_1 & a_2 \\
\emptyset & b_1 & b_2 \\
\emptyset & c_1 & c_2 \\
a_1 & 1 & 2 & b_1 & 9 & 10 & c_1 & 17 & 18 \\
a_1 & 3 & 4 & b_1 & 11 & 12 & c_1 & 19 & 20 \\
a_2 & 5 & 6 & b_2 & 13 & 14 & c_2 & 21 & 22 \\
a_2 & 7 & 8 & b_2 & 15 & 16 & c_2 & 23 & 24 \\
\end{array}
\]
There must be two pure blocks containing the treatment 1. By Lemma 2.5.1, one of the other two treatments in a pure block containing treatment 1 must occur in the b-group and the other in the c-group of mixed blocks. Since at this stage the sets within a group are combinatorially equivalent and the same holds true for the two numbered treatments within any mixed block, we may write the pure blocks containing treatment 1 as follows:

1  9  17
1  13  21.

From Lemma 2.5.2, it is seen that the two pure blocks containing treatment 2 cannot contain treatments 10, 14, 18, or 22. By Lemma 2.3.3, treatments 9, 13, 17, or 21 cannot appear in a pure block with 2. By Lemma 2.5.1, the pure blocks containing 2 must, therefore, contain four treatments from among 11, 12, 15, 16, 19, 20, 23 and 24. Since the elements within any of the pairs (11,12), (15,16), (19,20), and (23,24) are combinatorially equivalent, there is no loss in generality in assuming that treatments 11, 15, 19, and 23 appear in the pure blocks containing treatment 2. Due to Lemma 2.5.1, treatments 11 and 15 cannot appear together in the same pure block and neither can 19 and 23. Hence, there are just two possible arrangements of treatments 11, 15, 19, and 23 in the pure blocks containing 2:
First, let us assume that \((A)\) holds. We shall make use of the condition \(p_{11}^1 = p_{11}^2 = 1\), and for this purpose the following table of first associates is convenient.

<table>
<thead>
<tr>
<th></th>
<th>First Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta)</td>
<td>(a_1) (a_2) (b_1) (b_2) (c_1) (c_2)</td>
</tr>
<tr>
<td>(a_1)</td>
<td>0 (a_2) 1 2 3 4</td>
</tr>
<tr>
<td>(a_2)</td>
<td>0 (a_1) 5 6 7 8</td>
</tr>
<tr>
<td>(b_1)</td>
<td>0 (b_2) 9 10 11 12</td>
</tr>
<tr>
<td>(b_2)</td>
<td>0 (b_1) 13 14 15 16</td>
</tr>
<tr>
<td>(c_1)</td>
<td>0 (c_2) 17 18 19 20</td>
</tr>
<tr>
<td>(c_2)</td>
<td>0 (c_1) 21 22 23 24</td>
</tr>
<tr>
<td>1 (a_1)</td>
<td>2 9 17 13 21</td>
</tr>
<tr>
<td>2 (a_1)</td>
<td>1 11 19 15 23</td>
</tr>
<tr>
<td>9 (b_1)</td>
<td>10 1 17 * 24</td>
</tr>
<tr>
<td>23 (c_2)</td>
<td>24 2 15 * 10</td>
</tr>
</tbody>
</table>

Comparing the first associates of treatments 9 and 23 with those of treatment \(a_2\), it is seen that one of the two remaining first associates of treatment 9 must be one of the treatments 5, 6, 7, or 8 and that the same is true for treatment 23. Since they are of no particular interest, we shall represent them by
asterisks in the above table. Comparing the first associates of 9 and 23 with those of $b_1$, $b_2$, $c_1$, and $c_2$, it is seen that the remaining first associate of treatment 9 must be one of the treatments $21$, $22$, $23$, or $24$, and that the remaining first associate of treatment 23 must be one of the numbers 9, 10, 11, or 12. Comparing the first associates of treatment 9 with those of 1 and 2, we see that the remaining first associate of 9 cannot be 21 or 23, and since 1 and 22 are first associates of 21, it cannot be 22 because 1 and 22 would then be first associates of both 9 and 21. Hence, $24$ must be the remaining first associate of 9. Likewise, by comparing the first associates of 23 with those of 1 and 2, it is seen that the remaining first associate of 23 cannot be 9 or 11. Comparing the first associates of 23 and 11, it is seen that the remaining first associate of 23 cannot be 12; hence, it must be 10. However, this leads to a contradiction for now treatments 10 and $24$ are first associates of both 9 and 23, violating the condition $p_{11}^2 = 1$. Hence, the design is impossible in the case (A).

Next, assume that (B) holds. As before, we form a table of first associates. Comparing the first associates of 9 to 24 inclusive, with those of $a_2$, it is seen that in each case one of the two remaining first associates of each of these numbered treatments must be one of the treatments 5, 6, 7, or 8. We take no particular interest in their values, but represent them by asterisks in the following table of first associates.
<table>
<thead>
<tr>
<th></th>
<th>a₁</th>
<th>a₂</th>
<th>b₁</th>
<th>b₂</th>
<th>c₁</th>
<th>c₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₁</td>
<td>ø</td>
<td>a₂</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>a₂</td>
<td>ø</td>
<td>a₁</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>b₁</td>
<td>ø</td>
<td>b₂</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>b₂</td>
<td>ø</td>
<td>b₁</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
</tr>
<tr>
<td>c₁</td>
<td>ø</td>
<td>c₂</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>c₂</td>
<td>ø</td>
<td>c₁</td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
</tr>
<tr>
<td>1</td>
<td>a₁</td>
<td>2</td>
<td>9</td>
<td>17</td>
<td>13</td>
<td>21</td>
</tr>
<tr>
<td>2</td>
<td>a₁</td>
<td>1</td>
<td>11</td>
<td>23</td>
<td>15</td>
<td>19</td>
</tr>
<tr>
<td>9</td>
<td>b₁</td>
<td>10</td>
<td>1</td>
<td>17</td>
<td>*</td>
<td>24</td>
</tr>
<tr>
<td>10</td>
<td>b₁</td>
<td>9</td>
<td>3</td>
<td>22</td>
<td>*</td>
<td>19</td>
</tr>
<tr>
<td>11</td>
<td>b₁</td>
<td>12</td>
<td>2</td>
<td>23</td>
<td>*</td>
<td>18</td>
</tr>
<tr>
<td>12</td>
<td>b₁</td>
<td>11</td>
<td>20</td>
<td>*</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>b₂</td>
<td>14</td>
<td>1</td>
<td>21</td>
<td>*</td>
<td>20</td>
</tr>
<tr>
<td>14</td>
<td>b₂</td>
<td>13</td>
<td></td>
<td></td>
<td>*</td>
<td>23</td>
</tr>
<tr>
<td>15</td>
<td>b₂</td>
<td>16</td>
<td>2</td>
<td>19</td>
<td>*</td>
<td>22</td>
</tr>
<tr>
<td>16</td>
<td>b₂</td>
<td>15</td>
<td></td>
<td></td>
<td>*</td>
<td>17</td>
</tr>
<tr>
<td>17</td>
<td>c₁</td>
<td>18</td>
<td>1</td>
<td>9</td>
<td>*</td>
<td>16</td>
</tr>
<tr>
<td>18</td>
<td>c₁</td>
<td>17</td>
<td></td>
<td></td>
<td>*</td>
<td>11</td>
</tr>
<tr>
<td>19</td>
<td>c₁</td>
<td>20</td>
<td>2</td>
<td>15</td>
<td>*</td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>c₁</td>
<td>19</td>
<td></td>
<td>12</td>
<td>*</td>
<td>13</td>
</tr>
<tr>
<td>21</td>
<td>c₂</td>
<td>22</td>
<td>1</td>
<td>13</td>
<td>*</td>
<td>12</td>
</tr>
<tr>
<td>22</td>
<td>c₂</td>
<td>21</td>
<td>3</td>
<td>10</td>
<td>*</td>
<td>15</td>
</tr>
<tr>
<td>23</td>
<td>c₂</td>
<td>24</td>
<td>2</td>
<td>11</td>
<td>*</td>
<td>14</td>
</tr>
<tr>
<td>24</td>
<td>c₂</td>
<td>23</td>
<td></td>
<td></td>
<td>*</td>
<td>2</td>
</tr>
</tbody>
</table>
Comparing the first associates of 9, 11, 13, 15, 17, 19, 21, and 23 with those of \( b_1, b_2, c_1, \) and \( c_2 \), it is seen that the remaining first associate of each of the numbered treatments 9, 11, 13, ..., 23 must be one of the numbers indicated in the table below.

\[
\begin{array}{c|cccc}
   & 21 & 22 & 23 & 24 \\
9 & 11 & 17 & 18 & 19 & 20 \\
11 & 17 & 18 & 19 & 20 \\
13 & 17 & 18 & 19 & 20 \\
15 & 21 & 22 & 23 & 24 \\
17 & 13 & 14 & 15 & 16 \\
19 & 9 & 10 & 11 & 12 \\
21 & 9 & 10 & 11 & 12 \\
23 & 13 & 14 & 15 & 16 \\
\end{array}
\]

Upon comparing the first associates of 9, 11, 13, 15, 17, 19, 21, and 23 with those of 1, 2, 9, 11, 13, 15, 17, 19, 21, and 23, it is seen that in each case three of the candidates for first associates of the respective treatments are ineligible. Hence, the following pairs must occur in the design:

\[(9, 24), (13, 20), (17, 16), (21, 12), (11, 18), (15, 22), (19, 10), (23, 14).\]

The first associates so determined are entered in the table of first associates in the column to the right of the asterisks.
Now, treatments 3 and 4 must each occur in two pure blocks. The three replications of treatments 1, 2, 5, 6, 7, 8, 9, 11, 13, 15, 17, 19, 21, and 23 have been accounted for. Treatments 10, 12, 14, 16, 18, 20, 22, and 24 have each appeared twice in the design. Hence, the four pure blocks containing treatments 3 and 4 must contain some arrangement of the even numbered treatments from 10 to 24 inclusive, with each appearing once. By Lemma 2.5.1, treatment 10 cannot be paired with 12, 14, or 16. Comparing the first associates of 10 and 9, we see that 10 cannot appear in a block with 18 or 20. Comparing first associates of 9 and 10, it is seen that 10 cannot occur in a block with 24. Hence, 10 must appear in a block with 22. Now, at this stage, treatments 3 and 4 are combinatorially equivalent. Hence, there is no loss in generality in writing block

\[ 3 \ 10 \ 22. \]

The new first associates are entered in the preceding table to the left of the column of asterisks and are underlined for convenience in reading.

By Lemma 2.5.1, treatment 12 cannot occur in a block with 10, 14, or 16. A glance at the first associates of treatments 9, 10, and 11 shows that 12 cannot appear in a block with 18, 22, or 24 due to the conditions \[ p_{11}^1 = p_{11}^2 = 1. \] Hence, 12 must occur with 20.
However, it is now noted that treatments 12 and 13 have common first associates 20 and 21, violating the condition $p_{11}^2 = 1$ and showing that the design is impossible in case (B). This completes the proof of the impossibility of Design No. 14.

6. A Construction for Design No. 12

Design No. 12 has parameters

$$\begin{align*}
\begin{cases}
  v = b = 19, & \lambda_1 = 1, \ n_1 = 6, \ P_1 = \begin{pmatrix} 1 & 4 \\ 8 \end{pmatrix}, \\
  r = k = 3, & \lambda_2 = 0, \ n_2 = 12, \ P_2 = \begin{pmatrix} 2 & 4 \\ 7 \end{pmatrix}.
\end{cases}
\end{align*}$$

(2.6.1)

Since $\lambda_1 = 1$, $\lambda_2 = 0$, and $p_{11}^1 = k-2$, Lemma 2.3.3 and its five corollaries apply to this design. Lemma 2.3.3 requires that any two treatments belonging to different blocks containing a common treatment must be second associates of each other. Corollary 2.3.1 states that the design must contain 12 mixed blocks, each containing one lettered and two numbered treatments, and that these blocks can be divided into three distinct groups, each group being composed of two sets and each set containing two blocks. Corollary 2.3.2 requires that the eight numbered treatments appearing in a group be distinct, and Corollary 2.3.3 requires that the 12 mixed blocks contain exactly two complete replications of the numbered
treatments and that the third replication of the numbered treatments appear in four pure blocks. Corollary 2.3.4 requires that two sets of blocks belonging to different groups intersect in just one numbered treatment. Corollary 2.3.5 states that a mixed block cannot intersect a set belonging to another group in more than one treatment.

We shall now prove the following lemma:

Lemma 2.6.1. Under the conditions on the parameters of Design No. 12, each set of blocks of each of the three groups must intersect each of the four pure blocks of numbered treatments in exactly one treatment.

Proof: Due to the conditions $\lambda_1 = 1$ and $\lambda_2 = 0$, a pure block cannot intersect the two blocks of a set in more than one treatment each. By Lemma 2.3.3, two numbered treatments appearing in different blocks of the same set must be second associates of each other, and hence, must appear in different pure blocks. Now, by Corollary 2.3.2, each set of blocks contains four distinct numbered treatments, and there are just four pure blocks giving a complete replication of numbered treatments. Hence, the four numbered treatments of a set must be distributed one to each of the four pure blocks. This proves the lemma.
The three $\Theta$-blocks of Design No. 12 are:

\[
\Theta \ a_1 \ a_2 \\
\Theta \ b_1 \ b_2 \\
\Theta \ c_1 \ c_2 
\]

The four pure blocks may be taken as

\[
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
10 & 11 & 12 \\
\end{array}
\]

At this stage of the construction, the treatments within any pure block are combinatorially equivalent. Hence, we may take the two mixed blocks of the $a_1$-set as

\[
\begin{array}{ccc}
a_1 & 1 & 4 \\
a_1 & 7 & 10 \\
\end{array}
\]

in accordance with Lemma 2.6.1.

In order to see easily the relationship of association between any pair of second associates of $\Theta$ (numbered treatments), we shall form a set of six lattice diagrams in which a cell marked with a check ($\checkmark$) indicates that the coordinates of the cell are first associates of each other and consequently occur together in a mixed block, and in which a cell marked with a cross ($\times$) indicates that
the coordinates of the cell are second associates of each other and cannot occur together in any block.

\[\begin{array}{ccc}
4 & 5 & 6 \\
1 & \checkmark & x & x \\
2 & x & \checkmark & x \\
3 & x & x & x \\
\end{array} \quad \begin{array}{ccc}
7 & 8 & 9 \\
1 & x & \checkmark & x \\
2 & x & x & x \\
3 & x & x & \checkmark \\
\end{array} \quad \begin{array}{ccc}
10 & 11 & 12 \\
1 & x & x & x \\
2 & \checkmark & x & x \\
3 & x & x & \checkmark \\
\end{array}\]

It should be noted that the 54 cells account for all pairs of numbered treatments which could possibly occur in the mixed blocks.

As the construction proceeds, we shall check the cells corresponding to the first associates and cross out cells corresponding to second associates. We start out by checking the cells corresponding to the pairs (1,4) and (7,10) in the \(a_1\)-set. Lemma 2.3.3 precludes the occurrence of pairs (2,4), (3,4), (1,5), (1,6), (8,10), (9,10), (7,11), (7,12), (1,7), (1,10), (4,7), and (4,10); the corresponding cells are crossed out.

By Corollaries 2.3.3 and 2.3.4, the four treatments 1, 4, 7, and 10 must be distributed one to each of the \(b_1\)-, \(b_2\)-, \(c_1\)-, and \(c_2\)-sets. Hence, for each of the four numbered treatments 1, 4,
7, and 10 we need to determine one more lettered and one more numbered first associate. We let these four lettered first associates be represented by x, y, z, and w, remembering that they represent different first associates of θ. (They are b₁, b₂, c₁, and c₂ in some order or other.) We form a table showing the first associates of treatments 1, 4, 7, and 10. The remaining numbered first associates of 1, 4, 7, and 10 will be entered in the last column of the table as they are selected; they are underlined to distinguish them from the first associates which are known at the beginning.

<table>
<thead>
<tr>
<th>First Associates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

From the lattice diagrams, it is seen that the remaining numbered first associate of 1 must be one of the numbers 8, 9, 11, or 12. At this stage of the construction, with respect to treatments 1 and 4, treatments 8, 9, 11, and 12 are combinatorially equivalent. Hence, we may choose 8 as the remaining first associate of 1. By Lemma 2.3.3, the occurrence of the pairs (1, 9), (2, 8), (3, 8), and (4, 8) is ruled out.
The lattice diagrams now show that the remaining first associate of 7 must be one of the numbers 2, 3, 5, or 6. However, 1 and 7 are second associates of each other and have the common first associates $a_1$ and 8. Since $p^2_{11} = 2$, they cannot have any other common first associate; hence, neither 2 nor 3 can be a first associate of 7. Since treatments 5 and 6 are combinatorially equivalent at this stage of construction, we may choose 5 as the remaining numbered first associate of 7. This rules out the occurrence of pairs (5,8), (5,9), (4,7), (6,7), and (5,10).

From the lattice diagrams, it is now seen that the remaining numbered first associate of 4 must be one of the numbers 9, 11, or 12. Since 4 and 7 are second associates, it cannot be 9. Since treatments 11 and 12 are combinatorially equivalent at this stage of construction, we may choose 11 as the remaining numbered first associate of 4. The occurrence of the pairs (4,10), (4,12), (5,11), (6,11), and (1,11) is now ruled out.

The lattice diagrams now show that the remaining numbered first associate of 10 must be one of the numbers 2, 3, or 6. The relationship of association between 4 and 10 rules out 6. Combinatorially, 2 and 3 are equivalent; hence, we may choose 2 as the remaining numbered first associate of 10. Pairs (1,10), (3,10), (2,11), (2,12), and (2,7) are now ruled out.
We now recall that with respect to $\Theta$ there are four blocks containing a complete replication of the second associates of $\Theta$. Since $\Theta$ represents any treatment of the design, the same is true for each of the treatments 1, 4, 7, and 10. The complete replication of the second associates of treatment 1 must be

$$
\begin{array}{ccc}
\Theta & * & * \\
z & 5 & 7 \\
10 & 11 & 12 \\
6 & 9 & * \\
\end{array}
$$

where the asterisks represent lettered treatments. Similarly, the four blocks containing the second associates of each of 4, 7, and 10 are:

<table>
<thead>
<tr>
<th>2-nd Assoc. of 4</th>
<th>2-nd Assoc. of 7</th>
<th>2-nd Assoc. of 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta$ * * *</td>
<td>$\Theta$ * * *</td>
<td>$\Theta$ * * *</td>
</tr>
<tr>
<td>w 2 10</td>
<td>y 4 11</td>
<td>x 1 8</td>
</tr>
<tr>
<td>7 8 9</td>
<td>1 2 3</td>
<td>4 5 6</td>
</tr>
<tr>
<td>3 12 *</td>
<td>6 12 *</td>
<td>3 9 *</td>
</tr>
</tbody>
</table>

We thus get four new pairs $(6,9)$, $(3,12)$, $(6,12)$, and $(3,9)$ which can be checked in the lattice diagrams. Lemma 2.3.3 now precludes the occurrence of pairs $(3,6)$, $(9,12)$, $(6,8)$, $(4,9)$, $(1,12)$, $(3,11)$, $(5,12)$, $(6,10)$, $(2,9)$, and $(3,7)$. The four pairs just obtained, together with the six pairs $(1,4)$, $(7,10)$, $(1,8)$,
(4,11), (5,7), and (2,10), give ten of the twelve pairs of numbered treatments in the mixed blocks. Since Corollary 2.3.3 requires exactly two complete replications of numbered treatments in the mixed blocks, the remaining pairs must be formed from the treatments 2, 5, 8, and 11. The lattice diagrams show that only the pairs (2,5) and (8,11) are possible. We now see that the occurrence of pairs (2,5) and (8,11) rules out pairs (3,5), (2,6), (8,12), and (9,11); other pairs are also ruled out, but they have all been crossed out of the lattice diagrams previously. The reader may now check to see that the set of twelve number-pairs so chosen has produced no inconsistencies with the conditions on the design. It remains to be seen whether or not the number-pairs can be assigned to the letters in such a way that the conditions on the design are satisfied.

It is now noticed that if the pairs (2,5) and (8,11) are used to form a set, Lemma 2.3.3 will not rule out any of the twelve number-pairs under consideration. Furthermore, if we assign pairs (2,5) and (8,11) to the \( a_2 \)-set, then Corollary 2.3.2 is satisfied.

Due to the fact that 3 and 6 are second associates of each other and already have common first associates 9 and 12, treatments 3 and 6 cannot have any common lettered first associates. Likewise, it is seen that treatments 9 and 12 cannot have any common lettered first associates. Now 3 and 8 are second associates of each other and have common first associates 1 and 9. Since
x is a first associate of 8, it cannot be a first associate of 3. Also, x has occurred with 8, and 7. 8, and 9 constitute a pure block. Hence, by Lemma 2.6.1, x cannot occur with 9. Therefore, x must occur in a block containing 6 and 12.

Since 3 and 10 are second associates of each other and have the common first associates 2 and 12, w cannot appear in the same block with 3. Since 6 and 10 are second associates and have only the common first associate 12, w must be a first associate of 6, and consequently, w must also be a first associate of 9. Now, comparing first associates of 9 and 12, it becomes clear that y and z must both occur with 3. Also, 4 and 12 are second associates of each other and have common first associates 6 and 11. Hence, y cannot occur with 12. Consequently, y must occur with the pair (3,9), and z must occur with the pair (3,12). The x-, y-, z-, and w-sets are now known to be

\[
\begin{align*}
\text{x} & : 1 \ 8 \\
\text{y} & : 3 \ 9 \\
\text{z} & : 3 \ 12 \\
\text{w} & : 2 \ 10
\end{align*}
\]

\[
\begin{align*}
\text{x} & : 6 \ 12 \\
\text{y} & : 4 \ 11 \\
\text{z} & : 5 \ 7 \\
\text{w} & : 6 \ 9
\end{align*}
\]

Since the x-set contains 6 and 12 which also appear in the z- and w-sets, it is clear that the x- and y-sets compose one group while the z- and w-sets compose the other. If we identify x with \(b_1\), y with \(b_2\), z with \(c_1\), and w with \(c_2\), then the 19 blocks of the design will be:
It is now possible to check the relationships of associations to see that the conditions on the $p_{jk}^i$ ($i, j, k = 1, 2$) are satisfied and also to verify that the conditions laid down by Lemma 2.3.3 and Corollaries 2.3.1 through 2.3.5 are satisfied.

The plan above also serves as the Association Scheme.
7. Summary

The results of Chapter II can be summarized in the following theorem:

**Theorem 2.7.1.** The class of symmetrical connected partially balanced incomplete block designs with two associate classes and three replications contains only nine combinatorially possible designs, namely,

<table>
<thead>
<tr>
<th>v=b</th>
<th>r=k</th>
<th>λ₁</th>
<th>λ₂</th>
<th>n₁</th>
<th>n₂</th>
<th>(P_{11}^1)</th>
<th>(P_{11}^2)</th>
</tr>
</thead>
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<tr>
<td>5</td>
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<td>1</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>1</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
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<td>1</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
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<td>6</td>
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<td>9</td>
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<td>1</td>
<td>0</td>
<td>6</td>
<td>12</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>
CHAPTER III

DERIVATION OF THE PARAMETERS OF PARTIALLY BALANCED
INCOMPLETE BLOCK DESIGNS WITH TWO ASSOCIATE
CLASSES AND \( k > r = 3 \)

1. Introduction

Nair has enumerated four series of partially balanced incomplete block designs involving only two complete replications of each treatment. Only two of these series of designs are of the type having two associate classes. Bose has completely solved the case of connected partially balanced incomplete designs with two associate classes and two replications. Except for trivial cases, he has shown that in addition to the two series obtained by Nair there is only one other series possible. Our object is to completely enumerate all possible connected partially balanced incomplete block designs with two associate classes involving three replications and to solve those designs for which \( r = 3 \leq k \leq 10 \). The case \( r = k = 3 \) has already been considered in Chapter II. In this chapter we shall generalize Bose's results to the class of partially balanced incomplete block designs with two associate classes characterized by \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \) but with \( r \) unspecified. For designs having \( r = 3 \) and \( \lambda_1 > \lambda_2 > 0 \) we shall demonstrate a method of obtaining
lower and upper bounds on the parameter b which makes possible an exhaustive enumeration of the designs belonging to the subclass under consideration. This method also applies to partially balanced designs of \( r > 3 \) replications.

Nair \(^{11}\) established a necessary condition for partially balanced designs having \( k > r \), and this condition was used by Bose in \(^3\). Nair's condition is of fundamental importance in obtaining an exhaustive enumeration of all partially balanced designs with two associate classes characterized by \( k > r > \lambda_1 > \lambda_2 \). This condition is stated below in the form of a theorem.

**Theorem.** For partially balanced incomplete block designs with \( m \) associate classes having \( k > r \),

\[
|E| = 0
\]

where

\[
E = (e_{ab}), \quad a, b = 1, 2, \ldots, m,
\]

is defined by

\[
e_{ab} = r^6_{ab} + \sum_{j=1}^{m} \lambda_j p_{aj} b - \lambda a n a,
\]

\[
\delta_{ab} = 1 \text{ if } b = a, \quad \delta_{ab} = 0 \text{ if } b \neq a.
\]

Nair's proof depends on the notion of recovery of inter-block information from the design. A purely combinatorial proof without
appeal to any extraneous considerations is due to Bose $\mathcal{A}$.

For the special case of partially balanced incomplete block designs with two associate classes having $k > r$, Nair's condition simplifies to

\[(3.1.1) \quad |E| = (r-\lambda_1)(r-\lambda_2) + (\lambda_1-\lambda_2) \left\{ \left( r-\lambda_1 \right) p_{12}^2 - \left( r-\lambda_2 \right) p_{12}^1 \right\} = 0.\]

2. **An Exhaustive Enumeration of Partially Balanced Incomplete Block Designs with Two Associate Classes and $k > r = 3$.**

2.1. Adopting the convention $\lambda_1 > \lambda_2$, we list all possible combinations of values of $\lambda_1$ and $\lambda_2$ as follows:

\[
\begin{align*}
(1) & \quad \lambda_1 = 3, \lambda_2 = 2 \\
(2) & \quad \lambda_1 = 3, \lambda_2 = 1 \\
(3) & \quad \lambda_1 = 3, \lambda_2 = 0
\end{align*}
\]

\[
\begin{align*}
(4) & \quad \lambda_1 = 2, \lambda_2 = 1 \\
(5) & \quad \lambda_1 = 2, \lambda_2 = 0 \\
(6) & \quad \lambda_1 = 1, \lambda_2 = 0.
\end{align*}
\]

2.2. Any design belonging to subclass (1), (2), or (3) must be of the singular group divisible type since Theorem 2.1.41 applies regardless of the value of $k$. By Corollary 2.1.41, the subclass (3) contains only disconnected designs. The designs
belonging to subclasses (1) and (2), since they are of the singular group divisible type, are obtainable from balanced incomplete block designs whose parameters are given by

\[(3.2.21) \quad v^* = m, \quad b^* = b, \quad r^* = r, \quad k^* = k/n, \quad \lambda^* = \lambda_2.\]

(See Th. 2, Subsection 1.3, Chapter I.)

For balanced incomplete block designs it is well known that

\[(3.2.22) \quad v^*r^* = b^*k^*, \quad \lambda^*(v^* - 1) = r^*(k^* - 1).\]

Also the inequalities

\[(3.2.23) \quad b^* \geq v^* (or \quad k^* \leq r^*), \quad \lambda^* < r^* < b^*\]

hold except in the trivial case when the design reduces to a randomized block design.

(1) Case \(k > r = \lambda_1 = 3, \lambda_2 = 2\). Since \(k^* \leq r^* = 3\), the only possible values \(k^*\) can assume are 1, 2, and 3. When \(k^* = 1\), \((3.2.22)\) yields \(b^* = r^* = 3\) which fails to satisfy \((3.2.23)\), and when \(k^* = 2\), we are led to an absurdity since \(v^*\) is non-integral. However, when \(k^* = 3\), \((3.2.22)\) determines the simple balanced incomplete block design

\[(3.2.24) \quad v^* = b^* = 4, \quad r^* = k^* = 3, \quad \lambda^* = 2\]
whose solution has the four blocks

(3.2.25) \((A, B, C), (A, B, D), (A, C, D),\) and \((B, C, D)\).

Replacing each of the four treatments of (3.2.25) with \(m = 4\) groups of \(t\) treatments each, we obtain the corresponding series of singular group divisible designs having

(3.2.26) \(v = 4t, b = 4, r = 3, k = 3t, \lambda_1 = 3, \lambda_2 = 2, m = 4, n = t,\)

where \(t = 2, 3, 4, \ldots \).

(2) Case \(k > r = \lambda_1 = 3, \lambda_2 = 2\). Here too, \(k^*\) can assume only the values 1, 2, and 3. When \(k^* = 1\), (3.2.22) gives \(r^* = b^* = 3\) which fails to satisfy (3.2.23). When \(k^* = 2\), we obtain the balanced incomplete block design

(3.2.27) \(v^* = 4, b^* = 6, r^* = 3, k^* = 2, \lambda^* = 1,\)

a simple unreduced design whose solution is

(3.2.28) \((A, B), (A, C), (A, D), (B, C), (B, D), (C, D)\).

Replacing each treatment of (3.2.28) by a group of \(t\) treatments, we obtain the corresponding series of singular group divisible designs having

(3.2.29) \(v = 4t, b = 6, r = 3, k = 2t, \lambda_1 = 3, \lambda_2 = 1, m = 4, n = t,\)

where \(t = 2, 3, 4, \ldots \).
When \( k^* = 3 \), we obtain from (3.2.22) the balanced incomplete block design

\[
(3.2.210) \quad v^* = b^* = 7, \quad r^* = k^* = 3, \quad \lambda^* = 1,
\]

which is a member of the second orthogonal series of Fisher and Yates \( L10^J \). The solution of (3.2.210) is

\[
(3.2.211) \quad \begin{cases} 
(A, B, C), (B, D, E), (D, C, F), (C, E, G), \\
(E, F, A), (F, G, B), (G, A, D).
\end{cases}
\]

Replacing each treatment of (3.2.211) with a group of \( t \) treatments, we obtain the corresponding series of singular group divisible designs having

\[
(3.2.212) \quad v = 7t, \quad b = 7, \quad r = 3, \quad k = 3t, \quad \lambda_1 = 3, \quad \lambda_2 = 1, \quad m = 7, \quad n = t,
\]

where \( t = 2, 3, 4, \ldots \).

The three series of singular group divisible designs given by (3.2.26), (3.2.29), and (3.2.212) contain all partially balanced designs with two associate classes having \( k > r = \lambda_1 = 3 \) and \( \lambda_2 = 1 \) and 2. These designs have been obtained by Bose, Bhattacharya, and Shrikhande \( L5^J \).

2.3. For the subclass (5) of designs characterized by

\[
(3.2.31) \quad k > r = 3, \quad \lambda_1 = 2, \quad \lambda_2 = 0,
\]
Nair's condition (3.1.1) becomes

\[(3.2.32) \quad p_{12}^2 = 3p_{12}^1 - 3/2,\]

which is absurd since both \(p_{12}^1\) and \(p_{12}^2\) must be non-negative integers. Hence, the subclass under consideration is empty.

2.4. We shall now obtain a general derivation of the parameters of partially balanced designs with two associate classes characterized by

\[k > r \geq 2, \lambda_1 = 1, \text{ and } \lambda_2 = 0.\]

To this end we now establish the following useful lemma:

**Lemma 3.2.41.** For any partially balanced incomplete block design with two associate classes and

\[k > r \geq 2, \lambda_1 = 1, \text{ and } \lambda_2 = 0\]

the inequality

\[(3.2.41) \quad p_{11}^1 \leq (k - 2) + (r - 1)^2\]

holds.

Let the treatments \(\Theta\) and \(\emptyset\) be first associates. Then they occur together in exactly one block which we shall denote by \(B(\Theta, \emptyset)\). There are \(k - 2\) other treatments in this block which are first associates of \(\Theta\) as well as \(\emptyset\). Since \(\lambda_1 = 1\), \(\Theta\) and \(\emptyset\)
cannot occur together in any other block. Denote the \( r - 1 \) blocks in which \( \Theta \) occurs (but not \( \emptyset \)) by \( B_i(\Theta) \), \( (i = 1, 2, \ldots, r-1) \), and similarly the \( r - 1 \) blocks in which \( \emptyset \) occurs (but not \( \Theta \)) by \( B_j(\emptyset) \), \( (j = 1, 2, \ldots, r-1) \). If a treatment does not occur in \( B(\Theta, \emptyset) \) but is a first associate of \( \Theta \) as well as \( \emptyset \), it must occur exactly once in the blocks \( B_i(\Theta) \), and exactly once in the blocks \( B_j(\emptyset) \). But the block \( B_j(\emptyset) \) cannot have more than one treatment in common with any of the blocks \( B_1(\Theta), B_2(\Theta), \ldots, B_{r-1}(\Theta) \). Hence, \( B_j(\emptyset) \) cannot contain more than \( r - 1 \) first associates of \( \emptyset \). This holds for \( j = 1, 2, \ldots, r - 1 \). Hence, there cannot exist more than \( (r - 1)^2 \) treatments which occur once among the blocks \( B_i(\Theta) \) and once among the blocks \( B_j(\emptyset) \). Hence, \( \pi_{11} \) cannot exceed \((k - 2) + (r - 1)^2\).

For designs of the class under consideration Nair's condition (3.1.1) simplifies to

\[
(3.2.42) \quad r \pi_{12}^1 - (r - 1) \pi_{12}^2 = r(r - 1).
\]

From (1.1.24) and (1.1.26) we obtain

\[
(3.2.43) \quad n_{1} \pi_{12}^1 + n_{2} \pi_{12}^2 = n_{1} n_{2}.
\]

Solving (3.2.42) and (3.2.43) simultaneously, we have

\[
(3.2.44) \quad \pi_{12}^1 = \frac{n_{2}(r-1)(r+n_{1})}{n_{1}(r-1) + n_{2}r}
\]

and
From (1.1.22) and (1.1.23) we get

\[(3.2.46)\]

\[n_1 = r(k - 1)\]

and

\[(3.2.47)\]

\[v = n_2 + 1 + r(k - 1).\]

From (1.1.21) and (3.2.47) it follows that

\[(3.2.48)\]

\[b = r^2 + \frac{r(n_2 - r + 1)}{k}.\]

Since both \(b\) and \(r^2\) must be integral, we set

\[(3.2.49)\]

\[r(n_2 - r + 1) = sk\]

where \(s\) is an integer. Then, from (3.2.49), (3.2.48), (1.1.21), (3.2.44), (3.2.45), and (3.2.46) we get

\[(3.2.410)\]

\[n_2 = \frac{sk}{r} + r - 1,\]

\[(3.2.411)\]

\[b = r^2 + s,\]

\[(3.2.412)\]

\[v = \frac{(r^2 + s)k}{r},\]

\[(3.2.413)\]

\[p_{12} = \frac{(r - 1)k - \frac{r(r-1)^2(k-1)}{s + r(r-1)}}{n_1(r-1) + n_2 r},\]

and
From the last two expressions it is seen that \( r(r-1)^2(k-1) \) and \( r^2(r-1)(k-1) \) must both be integral multiples of \( s + r(r-1) \), and hence, their difference must also be divisible by \( s + r(r-1) \). Hence, we introduce an auxiliary integral parameter, \( t \), defined by

\[
(3.2.415) \quad t = \frac{r(r-1)(k-1)}{s + r(r-1)}
\]

Then,

\[
(3.2.416) \quad s = \frac{1}{t} r(r-1)(k-t-1), \quad t \neq 0.
\]

From (3.2.410) through (3.2.414) and (3.2.416) it follows that

\[
(3.2.417) \quad v = \frac{k}{t} \left( (r-1)(k-1) + t \right),
\]

\[
(3.2.418) \quad b = \frac{r}{t} \left( (r-1)(k-1) + t \right),
\]

\[
(3.2.419) \quad n_2 = \frac{1}{t} (r-1)(k-1)(k-t),
\]

\[
(3.2.420) \quad p_{12}^2 = (r-1)(k-t),
\]

and

\[
(3.2.421) \quad p_{12}^2 = r(k-t-1).
\]
From (3.2.420), (3.2.421), (1.1.24), and (1.1.25) we get

\begin{align*}
(3.2.422) & \quad p_{11}^1 = t(r-l) + k - r - 1, \\
(3.2.423) & \quad p_{22}^1 = \frac{1}{t}(r-1)(k-t)(k-t-1), \\
(3.2.424) & \quad p_{11}^2 = rt, \\
\end{align*}

and

\begin{align*}
(3.2.425) & \quad p_{22}^2 = \frac{1}{t}(r-1)(k-1)(k-2t) + t(rt-k). \\
\end{align*}

Applying Lemma 3.2.41 and the inequality (2.1.22) to (3.2.422), we obtain

\begin{align*}
(3.2.426) & \quad 1 \leq t \leq r,
\end{align*}

a most useful set of bounds on the integral parameter \( t \). It is now seen that the divisor, \( s + r(r-1) \), in (3.2.422) produces no difficulty because

\begin{align*}
(3.2.427) & \quad (r-1)(k-1) \leq s + r(r-1) \leq r(r-1)(k-1), \quad k > r \geq 2.
\end{align*}
In summary, all partially balanced incomplete block designs with two associate classes belonging to the class

\[ k > r \geq 2, \lambda_1 = 1, \text{ and } \lambda_2 = 0 \]

are obtainable from

\[
\begin{align*}
\nu &= \frac{k}{t} \left\lceil \frac{(r-1)(k-1) + t}{r} \right\rceil, \quad r = r, \quad \lambda_1 = 1, \\
b &= \frac{r}{t} \left\lceil \frac{(r-1)(k-1) + t}{k} \right\rceil, \quad k = k, \quad \lambda_2 = 0, \\
n_1 &= r(k-1), \quad n_2 = \frac{1}{t} (r-1)(k-1)(k-t),
\end{align*}
\]

\((3.2.428)\)

\[
\begin{align*}
P_1 &= \begin{pmatrix}
(t-1)(r-1) + k - 2 & (r-1)(k-t) \\
\frac{1}{t} (r-1)(k-t)(k-t-1)
\end{pmatrix}, \\
P_2 &= \begin{pmatrix}
rt & r(k-t-1) \\
\frac{1}{t} \left\lceil (r-1)(k-1)(k-2t) + t(rt-k) \right\rceil
\end{pmatrix},
\end{align*}
\]

where

\[ 1 \leq t \leq r. \]

In the case of designs of three replications, there are only three series, one corresponding to each of the three permissible values of \(t\). Setting \(t = 1, 2, \) and \(3\) in \((3.2.428)\) we obtain
\[
\begin{align*}
\text{v} &= k(2k-1), \ r = 3, \ \lambda_1 = 1, \ n_1 = 3(k-1), \\
b &= 3(2k-1), \ k = k, \ \lambda_2 = 0, \ n_2 = 2(k-1)^2, \\
P_1 &= \begin{pmatrix} k-2 & 2(k-1) \\ 2(k-1)(k-2) \end{pmatrix}, \ P_2 &= \begin{pmatrix} 3 & 3(k-2) \\ 2k^2 - 7k + 7 \end{pmatrix}, \\
\text{v} &= k^2, \ r = 3, \ \lambda_1 = 1, \ n_1 = 3(k-1), \\
b &= 3k, \ k = k, \ \lambda_2 = 0, \ n_2 = (k-1)(k-2), \\
P_1 &= \begin{pmatrix} k & 2(k-2) \\ (k-2)(k-3) \end{pmatrix}, \ P_2 &= \begin{pmatrix} 6 & 3(k-3) \\ k^2 - 6k + 10 \end{pmatrix},
\end{align*}
\]

and

\[
\begin{align*}
\text{v} &= \frac{k}{3}(2k+1), \ r = 3, \ \lambda_1 = 1, \ n_1 = 3(k-1), \\
b &= 2k+1, \ k = k, \ \lambda_2 = 0, \ n_2 = \frac{2}{3}(k-1)(k-3), \\
P_1 &= \begin{pmatrix} k+2 & 2(k-3) \\ \frac{2}{3}(k-3)(k-4) \end{pmatrix}, \ P_2 &= \begin{pmatrix} 9 & 3(k-4) \\ \frac{1}{3}(2k^2 - 17k + 39) \end{pmatrix},
\end{align*}
\]

where in the last series \(k\) must be of the form \(3p\) or \(3p + 1\), \(p\) being an integer.
2.5. We shall now derive the parameters of all partially balanced incomplete block designs with two associate classes having

\[(3.2.51) \quad k > r = 3, \lambda_1 = 2, \text{ and } \lambda_2 = 1.\]

For designs belonging to this subclass Nair's condition \((3.1.1)\) becomes

\[(3.2.52) \quad 2p_{12}^1 - p_{12}^2 - 2 = 0.\]

From \((1.1.24)\) and \((1.1.26)\) we obtain

\[(3.2.53) \quad n_1p_{12}^1 + n_2p_{12}^2 = n_1n_2.\]

Solving \((3.2.52)\) and \((3.2.53)\) simultaneously, we obtain

\[(3.2.54) \quad p_{12}^1 = n_2(n_1+2)/(n_1+2n_2) = 1 + n_1(n_2-1)/(n_1+2n_2)\]

and

\[(3.2.55) \quad p_{12}^2 = 2n_1(n_2-1)/(n_1+2n_2).\]

From \((1.1.22)\) and \((1.1.23)\) we get

\[(3.2.56) \quad v = 3k - n_1 - 2,\]

and from \((3.2.54)\) and \((1.1.21)\) we obtain

\[(3.2.57) \quad b = 9 - \frac{3}{k}(n_1+2).\]
From the positive integral conditions on $b$, $n_1$, and $k$, it is seen that we may define the auxiliary parameter, $s$, by

$$(3.2.58) \quad s = \frac{3}{k}(n_1+2),$$

where $s$ must be a positive integer. Hence,

$$(3.2.59) \quad b = 9 - s$$

and

$$(3.2.510) \quad n_1 = \frac{sk}{3} - 2.$$ From (3.2.59) and (1.1.21)

$$(3.2.511) \quad v = \frac{k}{3}(9-s),$$

and from (3.2.510) and (1.1.22)

$$(3.2.512) \quad n_2 = \frac{k}{3}(9-2s) + 1.$$ From (3.2.54), (3.2.55), (3.2.510), and (3.2.512) we obtain

$$(3.2.513) \quad p^1_{12} = \frac{s \sqrt{k(9-2s)} + 3}{9(6-s)},$$

and

$$(3.2.514) \quad p^2_{12} = \frac{2(9-2s)(sk-6)}{9(6-s)}.$$ Since for a strictly partially balanced design, $n_1$ and $n_2$ must be positive integers, it is seen from (3.2.512) that $s \leq 9/2$. Since
s must also be a positive integer, we have

\begin{equation}
1 \leq s \leq 4, \quad 5 \leq b \leq 8.
\end{equation}

By use of the expressions (3.2.59) through (3.2.515), the relationships (1.1.24) and (1.1.25), and the integral conditions on all parameters of a design, we can now determine one series of designs corresponding to each of the four permissible values of s. These four series completely exhaust the designs of the subclass under consideration.

Corresponding to \( s = 1 \) is the series given by

\begin{equation}
\begin{aligned}
\nu &= 8(15t+2), \quad r = 3, \quad \lambda_1 = 2, \quad n_1 = 15t, \\
b &= 8, \quad k = 3(15t+2), \quad \lambda_2 = 1, \quad n_2 = 15(7t+1), \\
P_1 &= \begin{pmatrix} 2(4t-1) & 7t+1 \\ 14(7t+1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} t & 14t \\ 7(13t+2) \end{pmatrix},
\end{aligned}
\end{equation}

where \( t \geq 1 \).

When \( s = 2 \), we get the series

\begin{equation}
\begin{aligned}
\nu &= 7(6t+1), \quad r = 3, \quad \lambda_1 = 2, \quad n_1 = 12t, \\
b &= 7, \quad k = 3(6t+1), \quad \lambda_2 = 1, \quad n_2 = 6(5t+1), \\
P_1 &= \begin{pmatrix} 7t-2 & 5t+1 \\ 5(5t+1) \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2t & 10t \\ 5(4t+1) \end{pmatrix},
\end{aligned}
\end{equation}
where $t \geq 1$.

The designs corresponding to $s = 3$ are given by

$$
\begin{cases}
    v = 2(3t-1), & r = 3, \quad \lambda_1 = 2, \quad n_1 = 3(t-1), \\
    b = 6, & k = 3t-1, \quad \lambda_2 = 1, \quad n_2 = 3t,
\end{cases}
$$

(3.2.518)

$$
P_1 = \begin{pmatrix} 2(t-2) & \frac{t}{\lambda_2} \\ 2t & \frac{1}{\lambda_2} \end{pmatrix}, \quad P_2 = \begin{pmatrix} t-1 & 2(t-1) \\ t+1 & 1-t \end{pmatrix},
$$

where $t \geq 2$.

Corresponding to $s = 4$ is the series whose parameters are

$$
\begin{cases}
    v = 5(3t-1), & r = 3, \quad \lambda_1 = 2, \quad n_1 = 6(2t-1), \\
    b = 5, & k = 3(3t-1), \quad \lambda_2 = 1, \quad n_2 = 3t,
\end{cases}
$$

(3.2.519)

$$
P_1 = \begin{pmatrix} 10t-7 & 2t \\ t & 1-t \end{pmatrix}, \quad P_2 = \begin{pmatrix} 4(2t-1) & 2(2t-1) \\ 1-t & 1 \end{pmatrix},
$$

where $t = 1$ only.

2.6. In summary, all partially balanced incomplete block designs with two associate classes having $k > r = 3$ are given by ten series (3.2.26), (3.2.29), (3.2.212), (3.2.429), (3.2.430), (3.2.431), (3.2.516), (3.2.517), (3.2.518), and (3.2.519).
The parameters are shown below in tabular form.

TABLE IIA

PARAMETERS OF P.B.I.B. DESIGNS WITH TWO ASSOCIATE CLASSES AND \( k > r = 3 \) (GROUP DIVISIBLE TYPE)

<table>
<thead>
<tr>
<th>Series No.</th>
<th>v</th>
<th>b</th>
<th>k</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>m</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>4t</td>
<td>4</td>
<td>3t</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>t</td>
</tr>
<tr>
<td>II</td>
<td>4t</td>
<td>6</td>
<td>2t</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>t</td>
</tr>
<tr>
<td>III</td>
<td>7t</td>
<td>7</td>
<td>3t</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>t</td>
</tr>
</tbody>
</table>

TABLE IIB

PARAMETERS OF P.B.I.B. DESIGNS WITH TWO ASSOCIATE CLASSES AND \( k > r = 3 \) (NOT GROUP DIVISIBLE TYPES)

<table>
<thead>
<tr>
<th>Series No.</th>
<th>v</th>
<th>b</th>
<th>k</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( P_{11} )</th>
<th>( P_{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td>( k(2k-1) )</td>
<td>3(2k-1)</td>
<td>k</td>
<td>1</td>
<td>0</td>
<td>3(k-1)</td>
<td>2(k-1)^2</td>
<td>k-2</td>
<td>3</td>
</tr>
<tr>
<td>V</td>
<td>( k^2 )</td>
<td>3k</td>
<td>k</td>
<td>1</td>
<td>0</td>
<td>3(k-1)</td>
<td>(k-1)(k-2)</td>
<td>k</td>
<td>6</td>
</tr>
<tr>
<td>VI</td>
<td>( \frac{k}{3}(2k+1) )</td>
<td>2k+1</td>
<td>k</td>
<td>1</td>
<td>0</td>
<td>3(k-1)</td>
<td>( \frac{2}{3}(k-1)(k-3) )</td>
<td>k+2</td>
<td>9</td>
</tr>
<tr>
<td>VII</td>
<td>8(15t+2)</td>
<td>8</td>
<td>3(15t+2)</td>
<td>2</td>
<td>1</td>
<td>15t</td>
<td>15(7t+1)</td>
<td>2(4t-1)</td>
<td>t</td>
</tr>
<tr>
<td>VIII</td>
<td>7(6t+1)</td>
<td>7</td>
<td>3(6t+1)</td>
<td>2</td>
<td>1</td>
<td>12t</td>
<td>6(5t+1)</td>
<td>7t-2</td>
<td>2t</td>
</tr>
<tr>
<td>IX</td>
<td>2(3t-1)</td>
<td>6</td>
<td>3t-1</td>
<td>2</td>
<td>1</td>
<td>3(t-1)</td>
<td>3t</td>
<td>2(t-2)</td>
<td>t-1</td>
</tr>
<tr>
<td>X</td>
<td>5(3t-1)</td>
<td>5</td>
<td>3(3t-1)</td>
<td>2</td>
<td>1</td>
<td>6(2t-1)</td>
<td>3t</td>
<td>10t-7</td>
<td>4(2t-1)</td>
</tr>
</tbody>
</table>
CHAPTER IV

UNSYMMETRICAL PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS
WITH TWO ASSOCIATE CLASSES AND \( k > r = 3 \)

1. Introductory Remarks

The constructions of the singular group divisible designs given by Series I, II, and III of Table IIA were made clear in the process of obtaining the parameters of these series in Chapter III. As was previously mentioned, these designs have been solved in \( L^5 \).

The design corresponding to \( k=2 \) of Series IV of Table IIB is a group divisible design of the semi-regular type and has a trivial solution. The design corresponding to \( k=3 \) is a triangular type design whose solution may be found in \( L^7 \). Its solution may also be obtained by means of the finite projective geometry \( PG(3,2) \) as explained in Chapter I of this study. Designs of Series IV corresponding to \( k \geq 4 \) are new. We shall prove in Section 3 of this Chapter that the designs corresponding to \( k=4 \) and to \( k > 6 \) are combinatorially impossible, and shall give a construction of the design corresponding to \( k=5 \).

The designs of Series V of Table IIB are the well known lattice designs of three replications. For all integral values of \( k \geq 3 \) solutions exist. These solutions are obtained as
follows: Arrange the $k^2$ treatments in a square array. The $3k$
blocks of the design consist of the $k$ columns and the $k$ rows of
the array and the $k$ blocks whose treatments are elements of the
array corresponding to the same letter of a superimposed Latin
square.

Series VI has parameters given by

\[
\begin{align*}
\text{(4.1.1)} & \quad \left\{ \begin{array}{l}
v = \frac{k}{3}(2k+1), \quad r = 3, \quad \lambda_1 = 1, \quad n_1 = 3(k-1), \\
b = 2k+1, \quad k = k, \quad \lambda_2 = 0, \quad n_2 = \frac{2}{3}(k-1)(k-3), \\
P_1 = \left( \begin{array}{cc} k+2 & 2(k-3) \\ \frac{2}{3}(k-3)(k-4) \end{array} \right), \quad P_2 = \left( \begin{array}{cc} 9 & 3(k-4) \\ \frac{1}{3}(2k^2-17k+39) \end{array} \right),
\end{array} \right.
\end{align*}
\]

where $k$ must be of the form $3p$ or $3p+1$, $p$ being an integer.

Shrikhande \cite{17} has shown that the dual of a balanced
incomplete block design with parameters $v^*, b^*, r^*, k^*$, and $\lambda^* = 1$
will always be a partially balanced incomplete block design
belonging to the series of designs having parameters given by

\[
\begin{align*}
\text{(4.1.2)} & \quad \left\{ \begin{array}{l}
v = \frac{k}{r}(rk-k+1), \quad r = r, \quad \lambda_1 = 1, \quad n_1 = r(k-1), \\
b = rk-k+1, \quad k = k, \quad \lambda_2 = 0, \quad n_2 = \frac{1}{r}(r-1)(k-r)(k-1), \\
P_1 = \left( \begin{array}{cc} k-2+(r-1)^2 & (r-1)(k-r) \\ \frac{1}{r}(r-1)(k-r)(k-r-1) \end{array} \right), \\
P_2 = \left( \begin{array}{cc} r^2 & r(k-r) \\ (k-r)^2+2(r-1)\frac{k(k-l)}{r} \end{array} \right).
\end{array} \right.
\end{align*}
\]
Setting $r=3$ in (4.1.2) we obtain (4.1.1) as a special case. The
question now arises as to the existence of the balanced incomplete
block designs

$$(4.1.3) \quad v^* = 2k+1, \ b^* = \frac{k}{3}(2k+1), \ k^* = 3, \ r^* = k, \ \lambda^* = 1,$$

where $k$ is of the form $3p$ or $3p+1$, $p$ an integer, corresponding to
Series VI.

Reiss [14J has shown that for every $v^*$ of the form $6p+1$ or
$6p+3$ an arrangement of $v^*$ objects in triplets such that every
pair of objects occurs in exactly one triplet is possible. Such
an arrangement, which is known as a simple triple system or a
Steiner triple system, is nothing more than the balanced incomplete
block design (4.1.3). Thus, the results of Reiss together with
those of Shrikhande are sufficient to guarantee the existence of
solutions of the designs of Series VI as duals of the corresponding
balanced incomplete block designs given by (4.1.3).

In Series VII of Table IIB, $k \geq 51$. Since, in this study,
we are interested in the construction of those designs for which
$3 \leq k \leq 10$, Series VII contains no designs of interest. The
same is true of Series VIII in which $k \geq 21$.

The design of Series IX corresponding to $t=2$ was solved in
\[\text{[?]}\] by dualizing the balanced incomplete block design

$$(4.1.4) \quad v = 6, \ b = 10, \ r = 5, \ k = 3, \ \lambda = 2.$$
In \[8\] it is shown that this design is of the triangular doubly-linked blocks type. We shall show in Section 4 of this Chapter that the designs of Series IX corresponding to \( t \geq 3 \) are combinatorially impossible.

Series X of Table IIB contains only one design since the non-negative integral condition on \( p_{22}^2 \) requires that \( t \leq 1 \) while that on \( p_{11}^2 \) requires that \( t \geq 1 \). This design, whose parameters are

\[
\begin{align*}
\begin{cases}
  v = 10, & r = 3, & \lambda_1 = 2, & n_1 = 6, & p_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\
  b = 5, & k = 6, & \lambda_2 = 1, & n_2 = 3, & p_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix},
\end{cases}
\end{align*}
\]

(4.1.5)

is the complement of the known design

\[
\begin{align*}
\begin{cases}
  v = 10, & r = 2, & \lambda_1 = 1, & n_1 = 6, & p_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \\
  b = 5, & k = 4, & \lambda_2 = 0, & n_2 = 3, & p_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix},
\end{cases}
\end{align*}
\]

(4.1.6)

whose solution may be found in \([16]\). For the solution of (4.1.5) see the Appendix, Design Dl. The complement of a given design is one having the same number of varieties and the same number of blocks as the given design and its blocks consist of those treatments which do not occur in the corresponding blocks of the given design.
2. The Relationship of Duality Between Members of a Certain Two-Parameter Series of Partially Balanced Designs

2.1. If we put $t = 1$ in the series of designs given by (3.2.428) we obtain the two-parameter series of designs having

\[\begin{align*}
v &= k(r-1)(k-1) + 1, \quad r = r, \quad \lambda_1 = 1, \quad n_1 = r(k-1), \\
b &= r(r-1)(k-1) + 1, \quad k = k, \quad \lambda_0 = 0, \quad n_2 = (r-1)(k-1)^2,
\end{align*}\]

\[(4.2.11)\]

\[P_1 = \begin{pmatrix} k-2 & (r-1)(k-1) \\ (r-1)(k-1)(k-2) \end{pmatrix}, \quad P_2 = \begin{pmatrix} r & r(k-2) \\ k^2(r-1)-(3r-2)(k-1) \end{pmatrix}.
\]

If in (4.2.11) we set

\[(4.2.12) \quad r = k = s+1,\]

we obtain the parameters of the series of geometrical designs (1.3.11) of Chapter I as a special case.

It will be noted that Lemma 2.3.3 and Corollaries 2.3.1, 2.3.2, 2.3.3, 2.3.4, and 2.3.5 apply to designs given by (4.2.11), and that we are dealing with the special case in which

\[n_2 = (r-1)(k-1)^2.\]
2.2. Let D be a design with a known solution. Form a
design $D^\ast$ as follows: Let the treatments of D be the blocks of
$D^\ast$, and let the blocks of D be the treatments of $D^\ast$. The design
$D^\ast$ has been called the dual of design D. The process described
is sometimes referred to as inverting or dualizing design D.

In general, the dual of a partially balanced design is not
itself a partially balanced design. Inversion of a partially
balanced design D can result in a dual design $D^\ast$ having a different
number of associate classes than design D. Inversion of a partially
balanced design D may yield a completely balanced incomplete block
design.

We shall show that the existence of any unsymmetrical design
D belonging to the series (4.2.11), implies the existence of
another design $D^\ast$ belonging to the same series which is the dual
of D.

2.3. Let us assume the existence of a solution of some
design belonging to the series of designs (4.2.11) and examine
its dual. Let the parameters of the dual be distinguished by
the asterisk as superscript. Clearly,

$$\begin{align*}
v^* &= r\sqrt{(r-1)(k-1)+1}, \quad r^* = k, \\
b^* &= k\sqrt{(r-1)(k-1)+1}, \quad k^* = r.
\end{align*}$$

(4.2.31)
Consider the $r$ blocks of (4.2.11) containing treatment $\Theta$. These $r$ blocks containing $\Theta$ go over into $r$ treatments appearing in a single block of the dual design. Since no pair of $\Theta$-blocks of (4.2.11) contain another common treatment, in the dual no treatment-pair corresponding to a pair of blocks containing $\Theta$ can occur together in a block (excepting the one block corresponding to treatment $\Theta$). Hence,

$$(4.2.32) \quad \lambda_1^* = 1 \text{ and } \lambda_2^* = 0.$$ 

Arbitrarily choose a block of the original design and a treatment within this block. There are $r-1$ other blocks containing this treatment. The arbitrarily chosen treatment may be called $\Theta$ and the $r$ blocks containing $\Theta$ are then the $\Theta$-blocks. The arbitrarily chosen $\Theta$-block intersects each of the other $r-1$ $\Theta$-blocks in exactly one treatment. It also intersects each of the $(r-1)(k-1)$ blocks of one group in a single treatment, but it intersects no other blocks. Hence, in the dual

$$(4.2.33) \quad n_1^* = (r-1)+(r-1)(k-1) = k(r-1).$$

We shall call these $n_1^*$ blocks in the original design (treatments in the dual) the first associates of the arbitrarily chosen block. Since the arbitrarily chosen block fails to intersect any of the blocks in the other $r-1$ groups, in the dual

$$(4.2.34) \quad n_2^* = (r-1)^2(k-1).$$
We shall call these $n_2^*$ blocks the second associates of the arbitrarily chosen block.

Now, consider any pair of blocks which are first associates.

There is no loss in generality in assuming that they are the $i$-th and $j$-th blocks containing treatment $\Theta$ ($1 \neq j; 1 \leq i, j \leq r$).

The first associates of the $i$-th block containing $\Theta$ consist of all $\Theta$-blocks except the $i$-th, and all the blocks in the $i$-th group (the $i$-th group being the one whose lettered treatments occur in the $i$-th $\Theta$-block); the first associates of the $j$-th block containing $\Theta$ consist of all $\Theta$-blocks except the $j$-th, and all blocks in the $j$-th group. The blocks which are common to the first associates of the $i$-th and $j$-th $\Theta$-blocks are the other $r-2$ $\Theta$-blocks. Hence,

\[(4.2.35) \quad p_{11}^{1*} = r-2,\]

a constant for any particular design. Since the conditions of Lemma 1.2.1 are satisfied, it follows that $p_{12}^{1*}$, $p_{21}^{1*}$, and $p_{22}^{1*}$ are constants, and from (1.2.4) and (1.2.5) we obtain

\[(4.2.36) \quad p_{12}^{1*} = p_{21}^{1*} = (r-1)(k-1)\]

and

\[(4.2.37) \quad p_{22}^{1*} = (r-1)(r-2)(k-1).\]
Next, consider a pair of blocks which are second associates of each other, i.e., a pair of blocks containing no common treatment. There is no loss in generality if we take one of them to be the first $\Theta$-block and the other to be a block in the $j$-th group ($j \neq 1$). The first associates of the first $\Theta$-block are the $r-1$ other $\Theta$-blocks and all the blocks in the first group. The first associates of a block in the $j$-th group consist of the $r-2$ other blocks in the same set of the $j$-th group, one block (not the first) among the $\Theta$-blocks, and $k-1$ blocks in each of the $r$ groups excepting the $j$-th. Hence,

\[(4.2.38)\quad p_{11}^{*} = k,\]

a constant for a particular design. Thus, it is seen that the conditions of Lemma 1.2.2 are satisfied, and hence, $p_{12}^{*}$, $p_{21}^{*}$, and $p_{22}^{*}$ are constants. Furthermore, from (1.2.6) and (1.2.7) we obtain

\[(4.2.39)\quad p_{12}^{*} = p_{21}^{*} = k(r-2)\]

and

\[(4.2.310)\quad p_{22}^{*} = r^{2}(k-1)-(3k-2)(r-1).\]

In the original design, any pair of blocks which are first associates intersect in a single treatment ($X_{11}^{*} = 1$) while any
pair of blocks which are second associates fail to intersect at all \((\lambda_2^* = 0)\).

We have shown that the conditions (i), (ii), and (iii) stated in the definition of a partially balanced incomplete design with two associate classes (See Subsection 1.1 of Chapter I) are satisfied. Hence, the dual of design (4.2.11) is a partially balanced incomplete block design with two associate classes. Its parameters are

\[
\begin{align*}
\nu^* &= r^-(r-1)(k-1) + \frac{1}{r}, \\
\rho^* &= k, \\
\lambda_1^* &= l, \\
\lambda_2^* &= (r-1)(k-1) + 1, \\
n_1^* &= k(r-1), \\
n_2^* &= (k-1)(r-1)^2,
\end{align*}
\]

(4.2.311)

If in (4.2.311) we replace \(k\) with \(r^*\) and \(r\) with \(k^*\), it is seen that the dual has the same form as the original design (4.2.11) and hence, belongs to the same series. We have proved that the existence of the original design (4.2.11) implies the existence of (4.2.311) as the dual of (4.2.11). If we now assume the existence of (4.2.311), it can be shown that its dual exists and is given by (4.2.11). The argument is precisely the same as
that used in arriving at (4.2.311) with the exception that r and k are interchanged. We have thus proved the following theorem:

Theorem 4.2.31. Between corresponding designs of (4.2.11) and (4.2.311) there exists a relationship of duality.

The theorem of duality implies that if a design belonging to one of the series (4.2.11) or (4.2.311) is impossible, then the corresponding design of the other series is also impossible. Suppose that there exists a solution of a design of one of the series but that the corresponding design of the other series is impossible. Then by Theorem 4.2.31, upon inverting the design whose solution is known, we obtain the solution of the corresponding design of the other series. But this contradicts our assumption. Consequently, the first statement of this paragraph is true.

Actually, the series (4.2.11) and (4.2.311) are the same since we may obtain (4.2.311) from (4.2.11) by interchanging r and k. However, the correspondence between designs of (4.2.11) and (4.2.311) gives a means of pairing off those designs which are duals of each other. They exist or fail to exist simultaneously.
3. Constructions and Impossibility Proofs
for Designs of Series IV and Their Duals

3.1. Putting $k = 3$ in the two-parameter series of designs (4.2.11), we obtain the single-parameter series

$$\begin{align*}
v &= 3(2r-1), \quad r = r, \quad \lambda_1 = 1, \quad n_1 = 2r, \\
b &= r(2r-1), \quad k = 3, \quad \lambda_2 = 0, \quad n_2 = 4(r-1), \\
P_1 &= \begin{pmatrix} 1 & 2(r-1) \\ 2(r-1) & 2r-3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} r & r \\ 3r-5 & \end{pmatrix}.
\end{align*}$$

Putting $k = 3$ in the two-parameter series of designs (4.2.11), we obtain the single-parameter series

$$\begin{align*}
v^* &= r(2r-1), \quad r^* = 3, \quad \lambda_1^* = 1, \quad n_1^* = 3(r-1), \\
b^* &= 3(2r-1), \quad k^* = r, \quad \lambda_2^* = 0, \quad n_2^* = 2(r-1)^2, \\
P_1^* &= \begin{pmatrix} r-2 & 2(r-1) \\ 2(r-1)(r-2) & 2r^2-7r+7 \end{pmatrix}, \quad P_2^* = \begin{pmatrix} 3 & 3(r-2) \\ 3r-7 & \end{pmatrix}.
\end{align*}$$

which is Series IV of Table IIB of Chapter III.

It was seen in the preceding Section that corresponding designs of (4.3.11) and (4.3.12) are duals of each other and exist or fail to exist simultaneously. In the remainder of this section we shall show that the designs of series (4.3.11) corresponding to $r = 4$
and \( r \geq 6 \) are combinatorially impossible, and we shall give a construction of the design of series (4.3.11) corresponding to \( r = 5 \). From this, the construction or impossibility of the corresponding design of series (4.3.12) which is the same as Series IV will follow.

3.2. We shall now prove the impossibility of the design of series (4.3.11) corresponding to \( r = 4 \). This design has parameters

\[
\begin{align*}
\{ & v = 21, r = 4, \lambda_1 = 1, n_1 = 8, P_1 = \begin{pmatrix} 1 & 6 \\ 6 & 6 \end{pmatrix}, \\
& b = 28, k = 3, \lambda_2 = 0, n_2 = 12, P_2 = \begin{pmatrix} 4 & 4 \\ 7 & 7 \end{pmatrix}. 
\}
\]

Lemma 2.3.3 requires that any two treatments appearing in different blocks containing a common treatment must be second associates of each other.

Corollary 2.3.1 states that the design must have 24 mixed blocks which can be divided into 4 distinct groups, each group containing 2 sets, each set containing 3 blocks, and each block containing one lettered and two numbered treatments.

Corollary 2.3.2 requires that each group of mixed blocks contains one complete replication of the numbered treatments, while Corollary 2.3.3 states the obvious - the design contains no pure blocks of numbered treatments.

Corollary 2.3.4 requires that two sets of blocks belonging
to different groups intersect in 3 numbered treatments, and
Corollary 2.3.5 states that each mixed block must intersect each
set of another group in exactly one numbered treatment.

Using Lemma 2.3.3 and Corollaries 2.3.1 through 2.3.5, we
may write, without loss of generality, the three $G$-blocks, the
six blocks of the $a$-group, the three blocks of the $b_1$-set, and the
distribution of lettered treatments throughout the design as shown
below:

\[
\begin{array}{cccc}
\emptyset & a_1 & a_2 \\
\emptyset & b_1 & b_2 \\
\emptyset & c_1 & c_2 \\
\emptyset & d_1 & d_2 \\
\end{array}
\]

<table>
<thead>
<tr>
<th>a-Group</th>
<th>b-Group</th>
<th>c-Group</th>
<th>d-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_1 1 2</td>
<td>b_1 1 7</td>
<td>c_1 * *</td>
<td>d_1 * *</td>
</tr>
<tr>
<td>a_1 3 4</td>
<td>b_1 3 9</td>
<td>c_1 * *</td>
<td>d_1 * *</td>
</tr>
<tr>
<td>a_1 5 6</td>
<td>b_1 5 11</td>
<td>c_1 * *</td>
<td>d_1 * *</td>
</tr>
<tr>
<td>a_2 7 8</td>
<td>b_2 * *</td>
<td>c_2 * *</td>
<td>d_2 * *</td>
</tr>
<tr>
<td>a_2 9 10</td>
<td>b_2 * *</td>
<td>c_2 * *</td>
<td>d_2 * *</td>
</tr>
<tr>
<td>a_2 11 12</td>
<td>b_2 * *</td>
<td>c_2 * *</td>
<td>d_2 * *</td>
</tr>
</tbody>
</table>

where the asterisks represent numbered treatments.

By Corollaries 2.3.2, 2.3.4, and 2.3.5, it is seen that
treatments 2, 4, and 6 must each occur in different blocks of
the $b_2$-set and so must treatments 8, 10, and 12. Hence, to
start with we may take the $b_2$-set as

$$
\begin{align*}
  b_2 & 2 * \\
  b_2 & 4 * \\
  b_2 & 6 *
\end{align*}
$$

where the asterisks represent the numbered treatments 8, 10, and 12. We shall show later that the pairing of the numbered treatments of the $b_2$-set is uniquely determined and that in fact 2, 4, and 6 can occur with only 8, 10, and 12, respectively.

In view of Lemma 2.3.3, no number-pair formed from 1, 2, 3, 4, 5, and 6 can occur in any block of the $b$-, $c$-, and $d$-groups, the same being true of number-pairs formed from 7, 8, 9, 10, 11, and 12. Thus, we may represent all number-pairs which might conceivably occur in the $b$-, $c$-, and $d$-groups by a lattice diagram whose horizontal coordinates are 1, 2, ..., 6 and whose vertical coordinates are 7, 8, ..., 12. A check ($\checkmark$) placed in a cell will indicate that the coordinates of that cell appear as a number-pair in some mixed block, while a cross (X) in a cell will indicate that the treatments corresponding to the cell's coordinates are second associates and cannot occur together in a block.

By Lemma 2.3.3, the occurrence of the number-pair $(i, j)$, $i = 1, 2, \ldots, 6; j = 7, 8, \ldots, 12$, in a mixed block of the $b$-, $c$-, or $d$-group precludes the appearance in the design of a
number-pair whose corresponding cell lies in the same row or column of the subsquare of side 2 indicated by the double lines and containing the cell \((i,j)\). Thus, the occurrence of the number-pairs \((1,7)\), \((3,9)\), and \((5,11)\) in the \(b_1\)-set of blocks rules out the occurrence of the number-pairs \((2,7)\), \((1,8)\), \((4,9)\), \((3,10)\), \((6,11)\), and \((5,12)\) in any blocks of the design. Furthermore, by Corollary 2.3.5, none of the number-pairs \((1,9)\), \((1,11)\), \((3,7)\), \((3,11)\), \((5,7)\), and \((5,9)\) can occur in the design.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
7 & \checkmark & X & X & X & \\
8 & X & \checkmark & X & X & \\
9 & X & \checkmark & X & X & \\
10 & X & X & \checkmark & X & \\
11 & X & X & & \checkmark & X \\
12 & X & X & & & \checkmark \\
\end{array}
\]

By Corollary 2.3.2, treatments 7, 9, and 11 must each occur once in the c-group and once in the d-group. We form the following number-pair diagram containing all available number-pairs involving treatments 7, 9, and 11:

\[
\begin{array}{ccc}
2 & 4 & 6 \\
7 & - & - & (4,7) & (6,7) \\
9 & (2,9) & - & - & (6,9) \\
11 & (2,11) & (4,11) & - & - \\
\end{array}
\]
Since there are only two pairs involving each of the treatments 7, 9, and 11, all number-pairs in the above diagram must occur in the design. Using Lemma 2.3.3 and comparing the above number-pairs with those in the a-group, we see that the number-pairs (2,10), (2,12), (4,8), (4,12), (6,8), and (6,10) cannot appear in the design. We place crosses in the cells having these coordinates. It is now seen from the lattice diagram that the only available number-pairs involving only even numbers are (2,8), (4,10), and (6,12). Thus, it is seen that the number-pairs occurring in the $b_2$-set are uniquely determined. In fact, the $b_2$-set must be

$$
\begin{array}{ccc}
  b_2 & 2 & 8 \\
  b_2 & 4 & 10 \\
  b_2 & 6 & 12 \\
\end{array}
$$

The only available number-pairs involving treatments 8, 10, and 12, which must each occur just once in each of the c- and d-groups, are shown in the following number-pair diagram:

$$
\begin{array}{ccc}
  1 & 3 & 5 \\
  8 &  & (3,8) & (5,8) \\
  10 & (1,10) &  & (5,10) \\
  12 & (1,12) & (3,12) &  \\
\end{array}
$$
Two pairs may be defined to be set compatible if it is possible for them to occur in the same set without violating Lemma 2.3.3 and its corollaries.

Clearly, none of the number-pairs lying in the same row or in the same column of this number-pair diagram are set compatible, and the same is true for the number-pair diagram involving treatments 7, 9, and 11. Furthermore, no two number-pairs occupying unsymmetrical positions with respect to the main diagonal of the number-pair diagram are set compatible, since if they did occur in the same set, Lemma 2.3.3 would rule out the appearance in the design of one of the other number-pairs of the diagram.

Using Corollary 2.3.5 and comparing the number-pair \((4,7)\) with the number-pairs of the \(a\)- and \(b\)-groups, it is seen that treatments 1, 3, 8, and 10 cannot appear in the same set with the pair \((4,7)\). Hence, none of the pairs involving treatments 8, 10, or 12 can appear in the same set with \((4,7)\). Consequently, the only number-pair which is set compatible with \((4,7)\) is \((2,9)\). But a third number-pair is required to complete the set. Hence, the design is combinatorially impossible.

3.3. We shall now give a construction for the design of \((4,3,11)\) corresponding to \(r = 5\). Its parameters are
By use of Lemma 2.3.3 and its corollaries, we may write, without loss of generality, the five $Q$-blocks, the eight mixed blocks of the $a$-group, the four blocks of the $b_1$-set, and the distribution of the lettered treatments throughout the remaining mixed blocks as shown below. The placing of the underlined numbered treatments comes later.

### PLAN FOR DESIGN (4.3.31)

<table>
<thead>
<tr>
<th>a-Group</th>
<th>b-Group</th>
<th>c-Group</th>
<th>d-Group</th>
<th>e-Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$ 1 2</td>
<td>$b_1$ 1 9</td>
<td>$c_1$ 4 2</td>
<td>$d_1$ 6 9</td>
<td>$e_1$ 8 2</td>
</tr>
<tr>
<td>$a_1$ 3 4</td>
<td>$b_1$ 3 11</td>
<td>$c_1$ 2 13</td>
<td>$d_1$ 2 13</td>
<td>$e_1$ 2 15</td>
</tr>
<tr>
<td>$a_1$ 5 6</td>
<td>$b_1$ 5 13</td>
<td>$c_1$ 7 14</td>
<td>$d_1$ 7 12</td>
<td>$e_1$ 5 12</td>
</tr>
<tr>
<td>$a_1$ 7 8</td>
<td>$b_1$ 7 15</td>
<td>$c_1$ 5 16</td>
<td>$d_1$ 3 16</td>
<td>$e_1$ 3 11</td>
</tr>
<tr>
<td>$a_2$ 9 10</td>
<td>$b_2$ 2 10</td>
<td>$c_2$ 8 13</td>
<td>$d_2$ 8 11</td>
<td>$e_2$ 6 11</td>
</tr>
<tr>
<td>$a_2$ 11 12</td>
<td>$b_2$ 4 12</td>
<td>$c_2$ 6 15</td>
<td>$d_2$ 4 13</td>
<td>$e_2$ 4 13</td>
</tr>
<tr>
<td>$a_2$ 13 14</td>
<td>$b_2$ 6 14</td>
<td>$c_2$ 3 10</td>
<td>$d_2$ 5 10</td>
<td>$e_2$ 7 10</td>
</tr>
<tr>
<td>$a_2$ 15 16</td>
<td>$b_2$ 8 16</td>
<td>$c_2$ 1 12</td>
<td>$d_2$ 1 14</td>
<td>$e_2$ 1 16</td>
</tr>
</tbody>
</table>
As in the preceding Subsection, we form a lattice design whose coordinates give all conceivably possible number-pairs which might appear in the b-, c-, d-, and e-groups.

By use of Lemma 2.3.3, we see, upon comparing the number-pairs appearing in the $b_1$-set with those appearing in the a-group, that the number-pairs $(2,9), (1,10), (4,11), (3,12), (6,13), (5,14), (8,15), \text{ and } (7,16)$ cannot occur in the design. Also, Corollary 2.3.5 rules out the appearance in the design of number-pairs $(1,11), (1,13), (1,15), (3,9), (3,13), (3,15), (5,9), (5,11), (5,15), (7,9), (7,11), \text{ and } (7,13)$.

By Corollaries 2.3.2, 2.3.4, and 2.3.5, treatments 2, 4, 6, and 8 must occur in different blocks of the $b_2$-set, and the same is true of treatments 10, 12, 14, and 16. It will be shown later
that there exists a unique pairing off of these treatments in the \( b_2 \)-set.

Each of the treatments 9, 11, 13, and 15 must occur in each of the c-, d-, and e-groups by Corollary 2.3.2. From the lattice diagram, it is seen that the only number-pairs involving 9, 11, 13, and 15 are those shown in the following number-pair diagram:

\[
\begin{array}{cccc}
2 & 4 & 6 & 8 \\
9 & \_ & (4,9) & (6,9) & (8,9) \\
11 & (2,11) & \_ & (6,11) & (8,11) \\
13 & (2,13) & (4,13) & \_ & (8,13) \\
15 & (2,15) & (4,15) & (6,15) & \_ \\
\end{array}
\]

Since there are only three pairs involving each of the treatments 9, 11, 13, and 15, each of the number-pairs in the above number-pair diagram must occur in the design. Comparing these number-pairs with those occurring in \( a_1^- \), \( a_2^- \), and \( b_1^- \) sets of blocks, it is seen by Lemma 2.3.3 that the number-pairs (2,12), (2,14), (2,16), (4,10), (4,14), (4,16), (6,10), (6,12), (6,16), (8,10), (8,12), and (8,14) cannot appear in the design.

From the lattice diagram, it is seen that the only available number-pairs involving even numbers are (2,10), (4,12), (6,14), and (8,16). Thus, the number-pairs appearing in the \( b_2 \)-set are uniquely determined.
Examination of the lattice diagram now shows that the only available number-pairs involving treatments 10, 12, 14, and 16 are those shown in the following number-pair diagram:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-</td>
<td>-</td>
<td>(3,10)</td>
<td>(5,10)</td>
</tr>
<tr>
<td>12</td>
<td>(1,12)</td>
<td>-</td>
<td>(5,12)</td>
<td>(7,12)</td>
</tr>
<tr>
<td>14</td>
<td>(1,14)</td>
<td>(3,14)</td>
<td>-</td>
<td>(7,14)</td>
</tr>
<tr>
<td>16</td>
<td>(1,16)</td>
<td>(3,16)</td>
<td>(5,16)</td>
<td>-</td>
</tr>
</tbody>
</table>

Each of these number-pairs must obviously occur in the design since three groups are as yet unformed.

In each of the number-pair diagrams, it is clear that number-pairs lying in the same row or in the same column are not set compatible since \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \). Furthermore, any two unsymmetrically located pairs with respect to the main diagonal of the number-pair diagram are not set compatible since their occurrence in the same set would then, by Lemma 2.3.3, rule out the appearance in the design of the number-pair lying at the intersection of the row and column containing the two pairs in question. Thus, the four number-pairs appearing in each of the sets of the c-, d-, and e-groups must consist of symmetrically located number-pairs in the two number-pair diagrams.

Now the number-pairs \((4,9), (6,9), \) and \((8,9)\) must be distributed one in each of the last three groups by Corollary 2.3.2. Suppose
we arbitrarily form the three blocks

\[
\begin{align*}
  c_1 &\ 4\ 9 \\
  d_1 &\ 6\ 9 \\
  e_1 &\ 8\ 9
\end{align*}
\]

Then by the preceding paragraph, we must also form the blocks

\[
\begin{align*}
  c_1 &\ 2\ 11 \\
  d_1 &\ 2\ 13 \\
  e_1 &\ 2\ 15
\end{align*}
\]

The occurrence of the pairs (4,9) and (2,11) in the \(c_1\)-set precludes the occurrence in this set of pairs involving 1, 3, 10, and 12 by Corollary 2.3.5. Thus, the \(c_1\)-set must be completed with either (6,15) and (8,13) or with (5,16) and (7,14). Suppose we complete the \(c_1\)-set with (5,16) and (7,14). Then by Corollary 2.3.2, we may complete the \(c_2\)-set by use of pairs (8,13), (6,15), (3,10) and (1,12).

The occurrence of pairs (6,9) and (2,13) in the \(d_1\)-set precludes the occurrence in the same set of any pairs involving 1, 4, 5, 8, 10, 11, 14, and 15 by Corollary 2.3.5. Thus, it is seen that the only pairs which can occur with (6,9) and (2,13) in the \(d_1\)-set are (7,12) and (3,16). Then by Corollary 2.3.2, the \(d_2\)-set must contain pairs (8,11), (4,15), (5,10), and (1,14).
Since the $e_1$-set contains pairs $(8,9)$, and $(2,15)$, Corollary 2.3.5 rules out the occurrence in the $e_1$-set of any pair involving $1, 7, 10, 16, 4, 6, 11, \text{or } 13$. Thus, it is seen that only $(5,12)$ and $(3,14)$ are available. Then by Corollary 2.3.2, the pairs $(6,11), (4,13), (7,10)$, and $(1,16)$ must occur in the $e_2$-set. This completes the construction of design $(4,3.31)$. The reader can verify that the conditions set forth in Lemma 2.3.3 and in Corollaries 2.3.1 through 2.3.5 are satisfied. The Plan which also serves as an Association Scheme is given on Page 106.

3.4. Let us now consider any design of $(4,3.11)$ having $r \geq 6$. We write the $r$ $G$-blocks as follows:

\begin{align*}
\emptyset & \quad a_1 \quad a_2 \\
\emptyset & \quad b_1 \quad b_2 \\
\ldots & \quad \ldots \quad \ldots \\
\emptyset & \quad \ell_1 \quad \ell_2
\end{align*}

Each group consists of $2(r-1)$ blocks, the lettered treatments within a group being denoted by the same letter bearing the appropriate subscript. Each group contains two sets and each set contains $r-1$ blocks, the blocks within a set containing the same lettered treatment (Corollary 2.3.1). Since $n_2 = 4(r-1)$,
each group must contain exactly one complete replication of the numbered treatments (Corollary 2.3.2). Without loss of generality, we may write the first group containing lettered treatments \(a_1\) and \(a_2\) as follows:

\[
\begin{array}{c c c}
\text{a}_1 & 1 & 2 \\
\text{a}_1 & 3 & 4 \\
\text{a}_1 & 5 & 6 \\
\cdots & \cdots & \cdots \\
\text{a}_1 & 2r-3 & 2r-2 \\
\hline \\
\text{a}_2 & 2r-1 & 2r \\
\text{a}_2 & 2r+1 & 2r+2 \\
\text{a}_2 & 2r+3 & 2r+4 \\
\cdots & \cdots & \cdots \\
\text{a}_2 & 4r-5 & 4r-4 \\
\end{array}
\]

where the continued dotted line separates the two sets, and the first column of numbered treatments (cutting across the blocks) contains only the odd integers 1, 3, 5, \(\ldots\), \(4r-5\) while the second column of numbered treatments contains only the even integers 2, 4, 6, \(\ldots\), \(4r-4\).

There is no loss in generality in writing the \(b_1\)-set as follows:
where the $b_1$-set contains only the odd numbered second associates of $Q$ (Corollaries 2.3.4 and 2.3.5). Then, the $b_2$-set must contain all the even numbered second associates of $Q$ (Corollary 2.3.2). The treatments $2, 4, 6, \ldots, 2r-2$ must occur one in each block of the $b_2$-set, the same being true for the $r-1$ treatments $2r, 2r+2, 2r+4, \ldots, 4r-4$. We shall now show that the pairing off of the even numbered treatments in the $b_2$-set is uniquely determined.

Since no pair of numbered treatments appearing in the $a_1$- or $a_2$-set can occur together in any block of the $b$-, $c$-, $\ldots$, and $\lambda$-groups ($\lambda_1 = 1, \lambda_2 = 0$, and Lemma 2.3.3), we can form a lattice diagram having horizontal coordinates $1, 2, 3, \ldots, 2r-2$ and vertical coordinates $2r-1, 2r, 2r+1, \ldots, 4r-4$. The coordinates of the cells of the lattice diagram give all conceivably possible number-pairs which might occur in the $b$-, $c$-, $\ldots$, and $\lambda$-groups. The cells whose coordinates appear as a pair in the design are marked with a check ($\checkmark$), and the cells whose coordinates are ruled out by Lemma 2.3.3 or one of its corollaries are marked with a cross. The cells corresponding to the number-pairs appearing
in the $b_1$-set are those lying in the upper left-hand corner of the subsquares of side 2 lying along the main diagonal of the lattice diagram.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>2r-5</th>
<th>2r-4</th>
<th>2r-3</th>
<th>2r-2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2r-1</td>
<td>✓</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r</td>
<td>x</td>
<td>✓</td>
<td>x</td>
<td>x</td>
<td></td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r+1</td>
<td>x</td>
<td></td>
<td>✓</td>
<td>x</td>
<td>x</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r+2</td>
<td>x</td>
<td>x</td>
<td></td>
<td>✓</td>
<td>x</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r+3</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td>✓</td>
<td>x</td>
<td>...</td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2r+4</td>
<td></td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td>✓</td>
<td>...</td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4r-7</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4r-6</td>
<td>x</td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4r-5</td>
<td></td>
<td>x</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The number-pairs whose coordinates are the numbers occurring with $i$ in the $a_1$- and $b_1$-sets ($i = 1, 3, 5, \ldots, 2r-3$) are ruled out by Lemma 2.3.3 and so are the number-pairs whose coordinates are the numbers occurring with $j$ in the $a_2$- and $b_1$-sets ($j = 2r-1, 2r+1, 2r+3, \ldots, 4r-5$). The cells corresponding to these number-pairs lie in the upper right and lower left-hand corners of the
subsquares of side 2 lying along the main diagonal of the lattice diagram. Furthermore, Corollary 2.3.5 rules out the occurrence of the following number-pairs:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>2r-5</th>
<th>2r-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2r-1</td>
<td>- - -</td>
<td>(3,2r-1)</td>
<td>(5,2r-1)</td>
<td>...</td>
<td>(2r-5,2r-1)</td>
</tr>
<tr>
<td>2r+1</td>
<td>(1,2r+1)</td>
<td>- - -</td>
<td>(5,2r+1)</td>
<td>...</td>
<td>(2r-5,2r+1)</td>
</tr>
<tr>
<td>2r+3</td>
<td>(1,2r+3)</td>
<td>(3,2r+3)</td>
<td>- - -</td>
<td>...</td>
<td>(2r-5,2r+3)</td>
</tr>
</tbody>
</table>

The corresponding cells in the lattice diagram are those lying in the odd numbered columns and odd numbered rows with the exception of those lying along the main diagonal of the lattice diagram. In other words, they are all cells lying in the upper left-hand corner of the subsquares of side two with the exception of those lying along the main diagonal of the lattice diagram.

Treatments 2r-1, 2r+1, 2r+3, ..., 2r-5 must each occur in each of the remaining r-2 groups (Corollary 2.3.2). The available number-pairs involving these treatments are given in the following number-pair diagram.
Since there are only \( r-2 \) number-pairs involving each of the treatments \( 2r-1, 2r+1, 2r+3, \ldots, \) and \( 4r-5 \), each of the number-pairs in the above diagram must occur in the design.

Comparing these number-pairs with those in the \( a_2 \)-set and using Lemma 2.3.3, it is seen that the following number-pairs are ruled out:

\[
\begin{array}{cccccc}
2 & 4 & 6 & 2r-4 & 2r-2 \\
2r-1 & - & - & (4,2r-1) & (6,2r-1) & \cdots & (2r-4,2r-1) & (2r-2,2r-1) \\
2r+1 & (2,2r+1) & - & - & (6,2r+1) & \cdots & (2r-4,2r+1) & (2r-2,2r+1) \\
2r+3 & (2,2r+3) & (4,2r+3) & - & - & \cdots & (2r-4,2r+3) & (2r-2,2r+3) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
lr-7 & (2,lr-7) & (4,lr-7) & (6,lr-7) & - & - & - & (2r-2,lr-7) \\
lr-5 & (2,lr-5) & (4,lr-5) & (6,lr-5) & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]
Thus, we cross out the cells lying in the lower right-hand corner of all subsquares of side 2 except for those lying along the main diagonal of the lattice diagram.

Examination of the lattice diagram now shows that the only cells for which both coordinates are even numbers are those lying along the main diagonal, i.e., cells whose coordinates are

$$(2, 2r), (4, 2r+2), \ldots, (2r-2, 4r-4).$$

Thus, the pairing off of the numbered treatments appearing in the $b_2$-set is uniquely determined. The $r-1$ blocks of the $b_2$-set are:

\[
\begin{array}{ccc}
b_2 & 2 & 2r \\
b_2 & 4 & 2r+2 \\
b_2 & 6 & 2r+4 \\
\vdots & \vdots & \vdots \\
b_2 & 2r-2 & 4r-4
\end{array}
\]

The occurrence of the number-pairs in the $b_2$-set rules out only number-pairs which have been previously excluded.

Each of the treatments $2r, 2r+2, 2r+4, \ldots, 4r-4$ must occur in each of the remaining $r-2$ groups (Corollary 2.3.2). The only available number-pairs involving these treatments are given in the following number-pair diagram:
<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>2r-5</th>
<th>2r-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2r</td>
<td>-</td>
<td>(3,2r)</td>
<td>(5,2r)</td>
<td>...</td>
<td>(2r-5,2r)</td>
</tr>
<tr>
<td>2r+2</td>
<td>(1,2r+2)</td>
<td>-</td>
<td>(5,2r+2)</td>
<td>...</td>
<td>(2r-5,2r+2)</td>
</tr>
<tr>
<td>2r+4</td>
<td>(1,2r+4)</td>
<td>(3,2r+4)</td>
<td>-</td>
<td>...</td>
<td>(2r-5,2r+4)</td>
</tr>
</tbody>
</table>

... ... ... ... ... ... ... ...

| 4r-6 | (1,4r-6) | (3,4r-6) | (5,4r-6) | ... | ... | (2r-3,4r-6) |
| 4r-4 | (1,4r-4) | (3,4r-4) | (5,4r-4) | ... | ... | (2r-3,4r-4) |

Again, there are only \( r-2 \) pairs involving the treatments in question. Hence, each of the number-pairs in the preceding diagram must occur in the design. Furthermore, these number-pairs and those in the number-pair diagram involving treatments \( 2r-1, 2r+1, 2r+3, \ldots, 4r-5 \) are the only available number-pairs.

In each of the number-pair diagrams of available pairs, it is obvious that number-pairs lying in the same row or in the same column are not set compatible. Furthermore, the occurrence in any set of two unsymmetrically located number-pairs with respect to the main diagonal of the number-pair diagram would rule out the occurrence of the number-pair lying at the intersection of the rows and columns containing the two number-pairs in question (Corollary 2.3.5). Hence, only symmetrically located pairs with respect to the main diagonal of the number-pair diagram can possibly be set compatible.
Now each set contains \( r - 1 \geq 5 \) blocks and consequently requires at least 5 number-pairs since by hypothesis, \( r \geq 6 \). But we have available a maximum of 4 possibly set compatible pairs, two from each number-pair diagram. Hence, all designs of the series \((4.3.11)\) having \( r \geq 6 \) are combinatorially impossible.

We have proved that the designs of \((4.3.11)\) corresponding to \( r = 4 \) and to \( r \geq 6 \) are combinatorially impossible, and have given a construction for the design corresponding to \( r = 5 \). Solutions are already known for the designs corresponding to \( r = 2 \) and 3. Now, the designs given by \((4.3.12)\) are known to be duals of the designs of \((4.3.11)\). From Theorem 4.2.31, it follows that the designs of \((4.3.12)\) corresponding to \( k^* = r = 2, 3, \) and 5 have solutions, while those designs of \((4.3.12)\) corresponding to \( k^* = r = 4 \) and to \( k^* = r \geq 6 \) are combinatorially impossible.

For the solution of the design of \((4.3.12)\) corresponding to \( k^* = r = 5 \), see Design D5 of the Appendix.

We may summarize the results of Section 3 in the following theorem:

**Theorem 3.4.1.** (A) The series of designs \((4.3.11)\) having parameters

\[
v = 3(2r-1), \quad r = r \geq 2, \quad \lambda_1 = 1, \quad n_1 = 2r, \quad p_1 = \begin{pmatrix} 1 & 2(r-1) \\ 2(r-1) & \end{pmatrix},
\]

\[
b = r(2r-1), \quad k = 3, \quad \lambda_2 = 0, \quad n_2 = 4(r-1), \quad p_2 = \begin{pmatrix} r & r \\ 3r-5 \end{pmatrix}
\]
contains only three combinatorially possible designs:

1. the lattice design with \( r = 2 \) and \( k = 3 \),
2. the symmetrical design with \( r = 3 \),

and

3. the design corresponding to \( r = 5 \).

The designs corresponding to \( r = 4 \) and to \( r \geq 6 \) are combinatorially impossible.

(B) The series of designs (4.3.12) having parameters

\[
v = p(2p-1), \quad r = 3, \quad \lambda_1 = 1, \quad n_1 = 3(p-1),
\]

\[
b = 3(2p-1), \quad k = p \geq 2, \quad \lambda_2 = 0, \quad n_2 = 2(p-1)^2,
\]

\[
P_1 = \begin{pmatrix} p-2 & 2(p-1) \\ 2(p-1)(p-2) \end{pmatrix},
\]

\[
P_2 = \begin{pmatrix} 3 & 3(p-2) \\ 2p^2-7p+7 \end{pmatrix}
\]

has only three combinatorially possible designs, namely, those corresponding to \( p = 2, 3, \) and \( 5 \). These are the duals of the corresponding designs of (4.3.11). The designs corresponding to \( p = 4 \) and to \( p \geq 6 \) are combinatorially impossible.
4. The Designs of Series IX

Interchanging the \( \lambda \)'s, we may rewrite Series IX in the form

\[
\begin{align*}
\mathbf{v} &= 2(3t-1), \quad r = 3, \quad \lambda_1' = 2, \quad n_1' = 3t, \\
\mathbf{b} &= 6, \quad k = 3t-1, \quad \lambda_2' = 1, \quad n_2' = 3(t-1), \\
\mathbf{p}_1' &= \begin{pmatrix} t+1 & 2(t-1) \\ t-1 & 2(t-2) \end{pmatrix}, \quad \mathbf{p}_2' = \begin{pmatrix} 2t & t \\ 2(t-2) & 2(t-2) \end{pmatrix}
\end{align*}
\]

where \( t \geq 2 \).

We shall first prove the following lemma:

Lemma 4.4.1. In a partially balanced incomplete block design with two associate classes having \( b = 6 \), \( r = 3 \), \( k \geq 5 \), \( \lambda_1 = 1 \), and \( \lambda_2 = 2 \), no block can contain any treatment \( \Theta \) and more than three first associates of \( \Theta \). Hence, \( n_1 \leq 9 \).

Proof: Assume that the contrary condition holds and that a block of the design contains a treatment \( \Theta \) and four or more first associates of \( \Theta \). Call this block \( B_1 \). Treatment \( \Theta \) must appear in two other blocks since \( r = 3 \), and each of these blocks must be free from the first associates of \( \Theta \) appearing in block \( B_1 \) since \( \lambda_1 = 1 \). Call these blocks \( B_2 \) and \( B_3 \). Now each of the first associates of \( \Theta \) must appear in two of the remaining three blocks. Arbitrarily choose a first associate of \( \Theta \) appearing in \( B_1 \); call
it treatment $a$. Let the two blocks containing $a$ but not $\theta$ be called blocks $B_4$ and $B_5$, and let the remaining block be designated $B_6$. Since $r = 3$, $\lambda_1 = 1$, and $\lambda_2 = 2$, the remaining first associates of $\theta$ occurring in $B_1$ must all occur in $B_6$ and each of them must occur in either $B_4$ or $B_5$. Thus, if $B_1$ contains four or more first associates of $\theta$, a pair of first associates of $\theta$ occurs in $B_1$, $B_6$, and in either $B_4$ or $B_5$. But this contradicts the conditions $\lambda_1 = 1$ and $\lambda_2 = 2$, showing that no block of the design can contain a treatment $\theta$ and more than three first associates of $\theta$. Since there are only three $\theta$-blocks and $\lambda_1 = 1$, it follows that $n_1 \leq 9$.

The immediate consequence of Lemma 4.4.1 is that the designs of Series IX corresponding to $t > 3$ are combinatorially impossible.

The design of Series IX corresponding to $t = 2$ is the triangular doubly-linked blocks design No. 3 of Table IIA of Bose and Shimamoto's paper [8]. We shall now show that the design of Series IX corresponding to $t = 3$ is also impossible.

Putting $t = 3$ in (4.4.1), we obtain

\[
\begin{align*}
\begin{cases}
 v = 16, & r = 3, \quad \lambda'_1 = 1, \quad n'_1 = 9, \quad p'_1 = \binom{4}{2}, \\
 b = 6, & k = 8, \quad \lambda'_2 = 2, \quad n'_2 = 6, \quad p'_2 = \binom{6}{2}
\end{cases}
\end{align*}
\]
By Lemma 4.4.1, the 9 first associates of treatment θ, which we shall call a, b, c, d, e, f, g, h, and i, must occur three in each block containing θ. Let π represent any one of the second associates of θ. We may write the three θ-blocks as follows:

θ π * * * a b c
θ π * * * d e f
θ * * * g h i,

where the ten asterisks represent the two replications of the remaining second associates of θ. Consider treatment π and its relationship with treatments g, h, and i. In the three remaining blocks treatments g, h, and i must each be replicated twice under the restriction \( \lambda_1^1 = 1 \) and \( \lambda_2^1 = 2 \). Each of the remaining blocks must contain a pair formed from the treatments g, h, and i with no pair being repeated since repetition of a pair would violate the condition \( \lambda_2^2 = 2 \). These pairs are necessarily

\[(g,h), (g,i), \text{ and } (h,i).\]

Now treatment π must occur in one of the remaining three blocks and will consequently occur with one of the above pairs but will fail to occur with the other two. Regardless of which of the pairs π occurs with, one of the treatments g, h, or i fails to occur in a block with π, contradicting the conditions \( \lambda_1^1 = 1, \lambda_2^1 = 2 \). Thus, the design under consideration is combinatorially impossible.
We may summarize the results of this Section in the following theorem:

**Theorem 4.4.1.** All designs of the series

\[ v = 2(3t-1), \ r = 3, \ \lambda_1 = 1, \ n_1 = 3t, \]
\[ b = 6, \ k = 3t-1, \ \lambda_2 = 2, \ n_2 = 3(t-1), \]
\[ p_1' = \binom{t+1}{t-1} \binom{2(t-1)}{t-1}, \quad p_2' = \binom{2t}{2(t-2)} \]

corresponding to \( t \geq 3 \) are combinatorially impossible.
APPENDIX

For the convenience of the experimentalist, the parameters, efficiency factors, association schemes, field layout plans, and four computational constants of the six new designs appearing in this study and having \( v < 100 \) are given. Bose and Shimamoto [8] have effected a substantial simplification in the intra- and inter-block analyses of partially balanced incomplete block designs with two associate classes by the introduction of association schemes and four constants \((c_1, c_2, H, \text{ and } \Delta)\). The values of the four constants depend only upon the parameters of the design and may be computed before the experiment is performed. Further information regarding association schemes and the use of the constants may be found in [15] which contains a fully worked out example illustrating their use in the analysis.

One of the designs listed here has the triangular association scheme which was described briefly in Chapter I. The other five designs have what may be called the Simple Type of association scheme. For all designs having the latter type of association scheme, \( \lambda_1 = 1 \) and \( \lambda_2 = 0 \), i.e., two treatments appearing together in a block of the design are first associates, while treatments which do not appear in the same block are second associates. Thus, the field layout plan also serves as the association scheme.
DESIGN D1 (See Series X, Table IIIB, Chapter III)

\[ v = 10, \ r = 3, \ \lambda_1 = 2, \ n_1 = 6, \ P_1 = \binom{3}{2} \begin{pmatrix} 1 \end{pmatrix}, \]
\[ b = 5, \ k = 6, \ \lambda_2 = 1, \ n_2 = 3, \ P_2 = \binom{4}{2} \begin{pmatrix} 0 \end{pmatrix}, \]
\[ c_1 = 2/3, \ c_2 = 1/3, \ H = 11/2, \ \Delta = 15/2, \ E = 0.92 \]

Association Scheme: Triangular (Triply-linked blocks)

\[
\begin{array}{cccc}
* & 1 & 2 & 3 & 4 \\
1 & * & 5 & 6 & 7 \\
2 & 5 & * & 8 & 9 \\
3 & 6 & 8 & * & 10 \\
4 & 7 & 9 & 10 & *
\end{array}
\]

Plan:

\[
\begin{array}{cccccc}
5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 8 & 9 & 10 \\
1 & 3 & 4 & 6 & 7 & 10 \\
1 & 2 & 4 & 5 & 7 & 9 \\
1 & 2 & 3 & 5 & 6 & 8
\end{array}
\]
DESIGN D2 (See Table I, Chapter II)

\[ v = b = 19, \lambda_1 = 1, n_1 = 6, P_1 = \begin{pmatrix} 1 & 4 \\ 8 \end{pmatrix}, \]
\[ r = k = 3, \lambda_2 = 0, n_2 = 12, P_2 = \begin{pmatrix} 2 & 4 \\ 7 \end{pmatrix}, \]
\[ c_1 = 3/19, c_2 = -6/19, H = 13/3, \Lambda = 38/9, E = 0.63 \]

Association Scheme: Simple

Plan:

\begin{align*}
1 & \quad 2 & \quad 3 \\
1 & \quad 4 & \quad 5 \\
1 & \quad 6 & \quad 7 \\
2 & \quad 8 & \quad 11 & \quad 4 & \quad 8 & \quad 15 & \quad 6 & \quad 12 & \quad 14 \\
2 & \quad 14 & \quad 17 & \quad 4 & \quad 13 & \quad 19 & \quad 6 & \quad 10 & \quad 19 \\
3 & \quad 9 & \quad 12 & \quad 5 & \quad 11 & \quad 18 & \quad 7 & \quad 9 & \quad 17 \\
3 & \quad 15 & \quad 18 & \quad 5 & \quad 10 & \quad 16 & \quad 7 & \quad 13 & \quad 16 \\
8 & \quad 9 & \quad 10 \\
11 & \quad 12 & \quad 13 \\
14 & \quad 15 & \quad 16 \\
17 & \quad 18 & \quad 19
\end{align*}
DESIGN D3 (See Series (4.3.11))

\[ v = 27, r = 5, \lambda_1 = 1, n_1 = 10, P_1 = \begin{pmatrix} 1 & 8 \\ 8 & \end{pmatrix}, \]

\[ b = 45, k = 3, \lambda_2 = 0, n_2 = 16, P_2 = \begin{pmatrix} 5 & 5 \\ 10 & \end{pmatrix}, \]

\[ c_1 = 1/9, c_2 = -1/9, H = 8, \Lambda = 15, E = 0.66 \]

**Association Scheme:** Simple

**Plan:**

```
1  2  3
1  4  5
1  6  7
1  8  9
1 10 11
```

```
2 12 13 4 12 20 6 20 15 8 20 17 10 20 19
2 14 15 4 14 22 6 22 13 8 22 13 10 26 13
2 16 17 4 16 24 6 16 27 8 18 23 10 16 23
2 18 19 4 18 26 6 18 25 8 14 27 10 14 25
3 20 21 5 13 21 7 24 19 9 22 19 11 22 17
3 22 23 5 15 23 7 17 26 9 15 26 11 15 24
3 24 25 5 17 25 7 14 21 9 16 21 11 18 21
3 26 27 5 19 27 7 12 23 9 12 25 11 12 27
```
DESIGN D4 (See Section 3, Chapter I)

\[ v = b = 40, \lambda_1 = 1, n_1 = 12, P_1 = \begin{pmatrix} 2 & 9 \\ 18 \end{pmatrix}, \]
\[ r = k = 4, \lambda_2 = 0, n_2 = 27, P_2 = \begin{pmatrix} 4 & 8 \\ 18 \end{pmatrix}, \]
\[ c_1 = 1/10, c_2 = -1/5, H = 13/2, \Delta = 10, E = 0.73 \]

Association Scheme: Simple

Plan:

\[
\begin{array}{cccc}
1 & 5 & 6 & 7 \\
1 & 14 & 15 & 16 \\
1 & 23 & 24 & 25 \\
1 & 32 & 33 & 34 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
5 & 2 & 8 & 11 & 14 & 2 & 17 & 20 \\
5 & 3 & 9 & 13 & 14 & 3 & 18 & 22 \\
5 & 4 & 10 & 12 & 14 & 4 & 19 & 21 \\
6 & 17 & 27 & 37 & 15 & 11 & 30 & 36 \\
6 & 18 & 28 & 35 & 15 & 12 & 31 & 35 \\
6 & 19 & 26 & 36 & 15 & 13 & 29 & 37 \\
7 & 20 & 31 & 39 & 16 & 8 & 28 & 40 \\
7 & 21 & 29 & 40 & 16 & 9 & 26 & 39 \\
7 & 22 & 30 & 38 & 16 & 10 & 27 & 38 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
32 & 2 & 35 & 38 & 23 & 2 & 26 & 29 \\
32 & 3 & 36 & 40 & 23 & 3 & 27 & 31 \\
32 & 4 & 37 & 39 & 23 & 4 & 28 & 30 \\
33 & 11 & 21 & 27 & 24 & 11 & 18 & 39 \\
33 & 12 & 22 & 26 & 24 & 12 & 17 & 40 \\
33 & 13 & 20 & 28 & 24 & 13 & 19 & 38 \\
33 & 14 & 19 & 31 & 34 & 8 & 19 & 31 \\
34 & 9 & 17 & 30 & 34 & 9 & 17 & 30 \\
34 & 10 & 18 & 29 & 34 & 10 & 18 & 29 \\
\end{array}
\]
DESIGN D5 (See Series IV, Table IIB, Chapter III)

\[ v = 45, \quad r = 3, \quad \lambda_1 = 1, \quad n_1 = 12, \quad \beta_1 = \begin{pmatrix} 3 & 8 \\ 24 \end{pmatrix}, \]

\[ b = 27, \quad k = 5, \quad \lambda_2 = 0, \quad n_2 = 32, \quad \beta_2 = \begin{pmatrix} 3 & 9 \\ 22 \end{pmatrix}, \]

\[ c_1 = 1/9, \quad c_2 = -1/3, \quad H = 24/5, \quad \Delta = 27/5, \quad \varepsilon = 0.77 \]

Association Scheme: Simple

Plan:

\[
\begin{array}{llllllllll}
1 & 2 & 3 & 4 & 5 \\
1 & 6 & 7 & 8 & 9 \\
1 & 10 & 11 & 12 & 13 \\
2 & 14 & 15 & 16 & 17 & 6 & 14 & 29 & 37 & 45 \\
2 & 18 & 19 & 20 & 21 & 6 & 18 & 23 & 31 & 39 \\
3 & 22 & 23 & 24 & 25 & 7 & 15 & 28 & 33 & 41 \\
3 & 26 & 27 & 28 & 29 & 7 & 19 & 22 & 35 & 43 \\
4 & 30 & 31 & 32 & 33 & 8 & 16 & 24 & 36 & 40 \\
4 & 34 & 35 & 36 & 37 & 8 & 20 & 27 & 30 & 42 \\
5 & 38 & 39 & 40 & 41 & 9 & 17 & 25 & 32 & 44 \\
5 & 42 & 43 & 44 & 45 & 9 & 21 & 26 & 34 & 38 \\
\end{array}
\]
DESIGN D6 (See Section 3, Chapter I)

\[ v = b = 85, \lambda_1 = 1, n_1 = 20, P_1 = \begin{pmatrix} 3 & 16 \\ 48 \end{pmatrix}, \]
\[ r = k = 5, \lambda_2 = 0, n_2 = 64, P_2 = \begin{pmatrix} 5 & 15 \\ 48 \end{pmatrix}, \]
\[ c_1 = 1/17, c_2 = -3/17, H = 42/5, \Delta = 17, E = 0.78 \]

Association Scheme: Simple

Plan:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 & 5 \\
1 & 6 & 7 & 8 & 9 \\
1 & 10 & 11 & 12 & 13 \\
1 & 14 & 15 & 16 & 17 \\
1 & 18 & 19 & 20 & 21 \\
\hline
2 & 22 & 23 & 24 & 25 \\
2 & 26 & 27 & 28 & 29 \\
2 & 30 & 31 & 32 & 33 \\
2 & 34 & 35 & 36 & 37 \\
3 & 38 & 39 & 40 & 41 \\
3 & 42 & 43 & 44 & 45 \\
3 & 46 & 47 & 48 & 49 \\
3 & 50 & 51 & 52 & 53 \\
\end{array}
\]

\[
\begin{array}{cccc}
6 & 22 & 38 & 54 & 70 \\
6 & 26 & 42 & 66 & 78 \\
6 & 30 & 46 & 58 & 82 \\
6 & 34 & 50 & 62 & 74 \\
7 & 23 & 39 & 57 & 72 \\
7 & 27 & 43 & 69 & 80 \\
7 & 31 & 47 & 61 & 84 \\
7 & 35 & 51 & 65 & 76 \\
\end{array}
\]

\[
\begin{array}{cccc}
10 & 22 & 43 & 63 & 83 \\
10 & 26 & 39 & 59 & 75 \\
10 & 30 & 51 & 67 & 71 \\
10 & 34 & 47 & 55 & 79 \\
11 & 23 & 42 & 64 & 85 \\
11 & 27 & 38 & 60 & 77 \\
11 & 31 & 50 & 68 & 73 \\
11 & 35 & 46 & 56 & 81 \\
\end{array}
\]
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BIBLIOGRAPHY


