ON THE DISTRIBUTION OF AVERAGES OVER THE VARIOUS LAGS OF CERTAIN STATISTICS RELATED TO THE SERIAL CORRELATION COEFFICIENTS*

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SUMMARY: The sum of all the different circular serial correlation coefficients defined in the usual manner, with lags of 1, 2, 3, ..., N-1 time units and a sample of N successive observations, turns out to be identically equal to -1 while the corresponding sum of the non-circular serial correlation coefficients, defined with the sum of squares of deviations from the mean as common denominator, is identically equal to -1/2. The customary definitions of the circular and non-circular serial correlation coefficients are slightly modified hereby by dropping the correction term due to the sample mean. It is shown in this note that a certain function of the average of these modified circular serial correlation coefficients and another function of the average of modified non-circular serial correlation coefficients based on a random sample of size N from a normal distribution with zero mean and a fixed variance have F-distributions with N-1 and 1 degrees of freedom.

Let $x_1, x_2, \ldots, x_N$ be a random sample of size N from a normal distribution with zero mean and unit variance.

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Define

\( f_k = \frac{\sum_{j=1}^{N} x_j x_{j+k}}{\sum_{j=1}^{N} x_j^2} \)

where \( x_j = x_{N+j} \), for all \( j \).

Let

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad \text{and}
\]

(2) \[ E_k = \frac{N}{\sum_{j=1}^{N} x_j x_{j+k}}. \]

Then one can see that

(4) \[ E_k = \frac{1}{2} x^T \left( \begin{pmatrix} 0^K + C^N-K + C^T K + C^T N-K \end{pmatrix} x \right), \]

where

\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{pmatrix}, \quad \text{and}
\]

\( C^T \) denotes the transpose of the matrix \( C \).

Now

(5) \[ \sum_{K=1}^{N} E_k = x^T \sum_{K=1}^{N-1} \left( \begin{pmatrix} C^K + C^T K \end{pmatrix} x \right) \]
But

\[
\sum_{K=1}^{N-1} (c^K + c^{-K}) = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix} = D \text{(say)}
\]

Therefore

\[
\sum_{K=1}^{N-1} E_K = X^T D X.
\]

The characteristic roots of the matrix \( D \) are easily seen to be \( N-1 \) and \( -1 \), the later repeated \( N-1 \) times.

But

\[
\sum_{K=1}^{N-1} r_K = \sum_{K=1}^{N-1} E_K = X^T D X.
\]

If \( Y_1, Y_2, \ldots, Y_N \) are related to \( X_1, X_2, \ldots, X_N \) by a certain orthogonal transformation we find from (8) that

\[
\sum_{K=1}^{N-1} r_K \text{ is distributed like }
\]

\[
(N-1) \frac{Y_1^2 - Y_2^2 - \ldots - Y_N^2}{Y_1^2 + Y_2^2 + \ldots + Y_N^2}.
\]
Now defining \( r_0 = 1 \), we get

\[
\sum_{K=0}^{N-1} r_k \text{ is distributed like } N \frac{\sum_{j=1}^{N} x_j^2}{\sum_{j=1}^{N} x_j^2 + \cdots + \sum_{j=1}^{N} y_N^2}
\]

where \( y_1, y_2, \ldots, y_N \) are normally and independently distributed variates with zero mean and a common variance.

Hence

\[
\bar{r} = \frac{\sum_{K=0}^{N-1} r_k}{N}
\]

is distributed like

\[
\frac{\sum_{j=1}^{N} x_j^2}{\sum_{j=1}^{N} x_j^2 + \cdots + \sum_{j=1}^{N} y_N^2}
\]

From (10) and (11) one can easily see that

\[
F = \frac{1 - \bar{r}}{(N-1) \bar{r}}
\]

has an F distribution with \( N-1 \) and 1 degrees of freedom.

Now, consider the non-circular serial correlation coefficient of lag \( K \) defined by

\[
r^x_k = \frac{\sum_{j=1}^{N-K} x_j x_{j+k}}{\sum_{j=1}^{N} x_j^2}
\]
It is easily seen that

\[(14) \quad E^*_k = \sum_{j=1}^{N-K} x_j x_{j+k} = \frac{1}{2} \begin{bmatrix} C^k + C^k \end{bmatrix} \begin{bmatrix} x \end{bmatrix}. \]

Therefore,

\[ N-1 \quad \sum_{K=1}^{N-1} r^*_K = \sum_{K=1}^{N-1} E^*_K / \sum_{j=1}^{N} x_j^2 : \]

\[ = \frac{1}{2} \begin{bmatrix} x^t \sum_{K=1}^{N-1} \begin{bmatrix} C^k + C^k \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \end{bmatrix} : \]

\[ = \frac{1}{2} \begin{bmatrix} x^t \begin{bmatrix} D x \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \end{bmatrix}. \]

Hence

\[(16) \quad r^* = \frac{N-1}{2} \begin{bmatrix} r \end{bmatrix} + \begin{bmatrix} N \end{bmatrix}, \]

is distributed like

\[(17) \quad \frac{1}{2N} + \frac{1}{2} \begin{bmatrix} \frac{Y_1^2}{Y_1^2 + Y_2^2 + \cdots + Y_N^2} \end{bmatrix}, \]

where \( Y_1, Y_2, \ldots, Y_N \) are normally independently distributed random variables with zero mean and a common variance.

From (16) and (17) one can easily see that

\[(18) \quad F = \frac{1}{N-1} \begin{bmatrix} \begin{bmatrix} N \end{bmatrix} - \begin{bmatrix} r^* \end{bmatrix} - 1 \end{bmatrix} \]
has an $F$-distribution with $N-1$ and 1 degrees of freedom. The function
$\frac{1}{2} F^{-\frac{1}{2}}$ of either (18) or (12) has the student distribution with $N-1$
degrees of freedom.

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