DISTRIBUTION OF THE LARGEST OR THE SMALLEST CHARACTERISTIC ROOT
UNDER NULL HYPOTHESIS CONCERNING COMPLEX MULTIVARIATE NORMAL POPULATIONS

by

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1. Introduction:

It has been pointed out by the author [1] that one can handle all the classical problems of point estimation and testing hypothesis concerning the parameters of complex multivariate normal populations in a similar manner as one handles those for multivariate normal populations in real variates. In [1, 2], the author has derived an asymptotic formulae for certain likelihood test-procedures and in [2], the author has mentioned the maximum characteristic root statistic for testing the reality of a covariance matrix. The distribution of the characteristic roots under null hypothesis established in those two papers can be written in a general form as

\[ c_1 \left( \prod_{j=1}^{q} \omega_j^m (1-\omega_j)^n \right) \left( \prod_{j=1}^{q} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 \right) \, d\omega_1 \ldots d\omega_q \]

where \( c_1 = \prod_{i=1}^{q} \frac{\Gamma(n + m + q + j)}{\Gamma(n+j)\Gamma(m+j)\Gamma(j)} \) and

\[ 0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_q \leq 1. \]

We may also note that when \( n \) is large, the joint distribution of \( n\omega_j = f_j \) (\( j = 1, 2, \ldots, q \)), \( 0 \leq f_1 \leq \ldots \leq f_q < \infty \), can be written as

\[ c_2 \left( \prod_{j=1}^{q} f_j^m \right) \exp\left( - \frac{1}{2} \sum_{j=1}^{q} f_j \right) \left( \prod_{j=1}^{q} \prod_{k=j+1}^{q} (f_j - f_k)^2 \right) \, df_1 \ldots df_k \]

where \( c_2 = \frac{1}{\prod_{j=1}^{q} \left[ \Gamma(m+j)\Gamma(j) \right]} \).

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In this paper, we derive the distribution of \( \omega_q \) (or \( f_q \)) and \( \omega_1 \) (or \( f_1 \)). The percentage points will be given and some applications will be discussed in another paper.

2. Distribution of \( \omega_q \) or \( \omega_1 \)

For the distribution of \( \omega_q \), we shall require the following two lemmas:

**Lemma 1:**

\[
\sum_{\mathcal{D}} \prod_{j=1}^{s} \left[ \int_{0}^{s} x_j^{m_j}(1-x_j)^{n_j} \, dx_j \right] = \prod_{j=1}^{s} \left[ \int_{0}^{1} x_j^{m_j}(1-x_j)^{n_j} \, dx_j \right]
\]

where \( \mathcal{D} : (0 \leq x_1 \leq \ldots \leq x_s \leq x), \ (x \leq 1) \); and on the left hand side \((m_1', n_1'), \ldots, (m_s', n_s')\) is any permutation of \((m_s', n_s'), \ldots, (m_1', n_1')\) and the summation is taken over all such permutations.

For proof, one may refer to Roy [3, (A.9.3), p. 203].

**Lemma 2:**

\[
\prod_{j=1}^{s} \prod_{k=j+1}^{s} (\omega_j - \omega_k)^2 = \sum \begin{vmatrix} \omega_{2q-2} & \omega_{2q-3} & \ldots & \omega_{q-1} \\ j_1 & j_2 & \ldots & j_q \end{vmatrix} \sum \begin{vmatrix} \omega_{2q-3} & \omega_{2q-4} & \ldots & \omega_{q-2} \\ j_1 & j_2 & \ldots & j_q \end{vmatrix} \ldots \sum \begin{vmatrix} \omega_{q-1} & \omega_{q-2} & \ldots & \omega_{q} \\ j_1 & j_2 & \ldots & j_q \end{vmatrix}
\]

where \( \Sigma \) means the summation over \((j_1', j_2', \ldots, j_q')\), the permutation of \((1, 2, \ldots, q)\), and \(|A|\) means the determinant of \(A\).

**Proof:** It is well known that a Vandermonde determinant

\[
\begin{vmatrix} \omega_1 & \omega_2 & \ldots & \omega_q \\ 1 & 2 & \ldots & q \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & q \end{vmatrix}^2 = \prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k) = \alpha, \ (say).
\]
Then, the above expression can be written as

\[ \alpha = \begin{vmatrix}
\sum_{j=1}^{q} \omega_j 2q-2 & \sum_{j=1}^{q} \omega_j 2q-3 & \cdots & \sum_{j=1}^{q} \omega_j q-1 \\
\sum_{j=1}^{q} \omega_j 2q-3 & \sum_{j=1}^{q} \omega_j 2q-4 & \cdots & \sum_{j=1}^{q} \omega_j q-2 \\
\cdots & \cdots & \cdots & \cdots \\
\sum_{j=1}^{q} \omega_j q-1 & \sum_{j=1}^{q} \omega_j q-2 & \cdots & q
\end{vmatrix} \]

If in the right hand side, any two \( j_i \) \( j_t \) are equal, then the value of the determinant is zero. Hence the summation over the right hand side over \( (j_1, j_2, \ldots, j_q) \) reduces to the permutations of \( (1, 2, \ldots, q) \), which establishes the lemma 2.

Now we shall prove the following theorem:

**Theorem 1:** If the joint distribution of \( \omega_1, \omega_2, \ldots, \omega_q \) is given by (1), then

\[ \Pr(\omega_q \leq x) = c_1 \begin{vmatrix}
\beta_0 & \beta_1 & \cdots & \beta_{q-1} \\
\beta_1 & \beta_2 & \cdots & \beta_q \\
\cdots & \cdots & \cdots & \cdots \\
\beta_{q-1} & \beta_q & \cdots & \beta_{2q-2}
\end{vmatrix} = c_1 |(\beta_{i+j-2})| \]
where \( c_1 \) is defined in (2), \( \beta_{i+j-2} = \int_0^x \omega^{i+j-2}(1-\omega)^n \, d\omega \)

for \( i, j = 1, 2, \ldots, q \) and \( \beta_{i+j-2} \) is a \( q \times q \) matrix.

**Proof:** By definition, we have

\[
\Pr(\omega_q \leq x) = \Pr(0 \leq \omega_1 \leq \ldots \leq \omega_q \leq x)
\]

\[
= c_1 \int_{\mathcal{D}} \prod_{j=1}^{m} \left[ \omega_j^{(1-\omega_j)^n} \right] \prod_{j=1}^{q-1} \prod_{k=j+1}^{q} (\omega_j - \omega_k)^2 \prod_{j=1}^{q} d\omega_j,
\]

where \( \mathcal{D} : (0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_q \leq x, x \leq 1) \).

Using lemma 2, the above expression can be written as

\[
(4) \quad \Pr(\omega_q \leq x) = c_1 \sum \begin{vmatrix}
\omega_1^{q-2} & \omega_2^{q-3} & \ldots & \omega_q^{q-1} \\
\omega_1^{q-3} & \omega_2^{q-4} & \ldots & \omega_q^{q-2} \\
& \ldots & \ldots & \ldots \\
\omega_1^{q-1} & \omega_2^{q-2} & \ldots & \omega_q^{q-0}
\end{vmatrix} \prod_{j=1}^{q} \left[ \omega_j^{m(1-\omega_j)^n} \, d\omega_j \right],
\]

where \( \sum \) means the summation over \((j_1, \ldots, j_q)\), the permutation of \((1, 2, \ldots, q)\). Now the determinant in the integral sign of \((4)\), can be written as

\[
\Sigma_1 \text{ sign } (t_1, \ldots, t_q) \omega_{j_1}^{q-1+t_1} \omega_{j_2}^{q-2+t_2} \ldots \omega_{j_q}^{t_q}
\]

where \((t_1, \ldots, t_q)\) is a permutation of \((0, 1, \ldots, q-1)\), \(\text{sign } (t_1, \ldots, t_q)\) is positive if the permutation is even and negative if the permutation is odd, and \(\Sigma_1\) means the summation over all such permutations. Then \((4)\) becomes
\[ \Pr(w_q \leq x) = c_1 \sum_{j_1} \cdots \sum_{j_q} \int \text{sign}(t_1, \ldots, t_q)(\omega_{j_1}^{q-1+t_1} \cdots \omega_{j_q}^t) \cdot \prod_{j=1}^q \left[ \omega_j^m (1-\omega_j)^n \right] \ d\omega_j. \]

First taking summation over \((j_1', j_2', \ldots, j_q')\), the permutation of \((1, 2, \ldots, q)\) and applying the lemma \(1\), we get

\[ \Pr(w_q \leq x) = c_1 \sum_{j_1} \cdots \sum_{j_q} \text{sign}(t_1, \ldots, t_q) \beta_{q-1+t_1} \beta_{q-2+t_2} \cdots \beta_t = c_1 |(\beta_{i+j-2})| \]

which proves the theorem \(1\).

It may be noted here that

\[ \Pr(w_1 \leq x) = 1 - \Pr(w_1 > x) = 1 - \Pr(x \leq w_1 \leq \ldots \leq w_q \leq 1). \]

Going back to the c.d.f. of \((\omega_1, \ldots, \omega_q)\) and using the transformation \(\omega_j = 1 - z_j (j = 1, 2, \ldots, q)\), we have

\[ (5) \quad \Pr(w_1 \leq x) = 1 - \Pr(x \leq w_1 \leq \ldots \leq w_q \leq 1) = 1 - c_1 |(\delta_{i+j-2})| \]

where \(\delta_{i+j-2} = \int_0^{1-x} z^{n+i+j-2} (1-z)^m \ dz\) and \((\delta_{i+j-2})\) is a \(qxq\) matrix.

**Theorem 2:** If the distribution of \(f_1, \ldots, f_q\) is given by \((2)\) then

\[ (6) \quad \Pr(f_q \leq x) = c_2 |(\gamma_{i+j-2})| \]

where \(\gamma_{i+j-2} = \int_0^x \omega^{m+i+j-2} \exp(-\omega) \ d\omega\), \((\gamma_{i+j-2})\) is a \(qxq\) matrix and \(c_2\) is defined in \((2)\).

Proof is similar to that of theorem \(1\).

3. The author thanks Professor S. N. Roy for discussion and help.
REFERENCES

