BINNARY RANDOM WALK ON A SEMI-INFINITE LINE WITH ONLY
AN UPPER VARIABLE ABSORBING BARRIER

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Summary

Some statistical problems may be described by a binary random walk scheme on a semi-infinite line limited by an upper absorbing barrier variable with the number n of steps.

If the absorbing barrier changes linearly known synthetical methods may be applied directly giving the exact value (or optimal approximation) of the probability of absorption P and average number \( \bar{n} \) of steps leading to the absorption (if \( P=1 \)). These results allow to give an extensive sufficient condition to be \( P=1 \) when the absorbing barrier is not changing linearly; in this case the problem of the distribution of the number n of steps leading to the absorption must be investigated to estimate P (if \( P < 1 \)) or \( \bar{n} \) (if \( P=1 \)).

Any binary random walk scheme can be transformed into an equivalent scheme whose binary system of displacements at each step of the random walk is (0,1). By using this system of displacements, if the corresponding absorbing barrier is not decreasing with n, a recurrent expression of the absorption probability \( P(n) \) after n steps exists as function of some preceding probabilities \( P(m) \), \( m < n \).
CHAPTER I (General Preliminaries)

I.1 - There are statistical problems which:

i) consider sequences \( \{ O_n \} \) of independent trials, or steps, or stages, or experiments (in general operations \( O_n \)) in each of which one of two necessary and incompatible quantitative events, either \( x' \) or \( x'' \), happens.

If \( x_n \) is the \( n \).th value of the sequence \( \{ x_n \} \) associated to the sequence \( \{ O_n \} \) \( (O_n \to x_n) \), it is a realization of the r.v. \((^o)x_n = x_n \) (independent of \( n \)) which assumes the value \( x_n = x' \) with probability \( q \) and the value \( x_n = x'' \) with probability \( p \ (x' < x'', \ p+q=1) \);

ii) are concerned with the sequence \( \{ x(n) \} \) of the partial series

\[ x(n) = \sum_{i=1}^{n} x_i \]

associated to the arrangement of \( x' \) and \( x'' \) in a same sequence \( \{ x_1, x_2, \ldots, x_n, \ldots \} \);

iii) define

1) for each \( n \) an event \( E_n \subset \mathbb{R}^n \) which happens when for the first time the sequence \( \{ x(i) \} \) exactly at the \( n \).th operation \( O_n \) (and not before for any \( i < n \)) assumes a value \( x(n) = x'(n) \) which is equal or greater of a given boundary \( A(n) > nx' \) depending on \( n \):

\[ (1) \quad (x_1, x_2, \ldots, x_n) \in E_n \iff \begin{cases} x(i) < A(i) \quad \text{for each } i=1,2,\ldots,n-1 \\ x(n) = x'(n) \geq A(n) \end{cases} \]

2) the total event \( E = \bigcup_{n=1}^{\infty} E_n \);  

3) the event \( \overline{E} \) contrary to \( E \):

\[ (x_1, x_2, \ldots, x_n, \ldots) \in \overline{E} \iff x(n) < A(n) \quad \text{for each } n=1,2,\ldots,\infty \implies \overline{E} \subset \mathbb{R}^\infty ; \]

(By definition the events \( E_n \), \( n=1,2,\ldots,\infty \), and \( \overline{E} \) are necessary and incompatible: some one of these events can be empty);

\( (^o) \) We will indicate the r.v.'s between inverted commas : "(r.v)".
iv) stop the sequence \( \{0_n\} \) at the \( n \)-th operation \( 0_n \) if the event \( E_n \) occurs;

v) aim at estimating the probability \( P = pr(E) = \sum_{n=1}^{\infty} pr(E_n) \) that an event \( E_n \in E \) will occur for any finite \( n \): \( pr(E \cup E) = pr(E) + pr(E) = 1 \);

vi) define the r.v. \( \nu_n \), which assume the values \( n = 1, 2, \ldots, \infty \) with probability

\[
P(n) = pr(\nu_n = n) = pr(E_n) \]

for each finite \( n \) (necessarily

\[
P = n \sum_{n=1}^{\infty} P(n) = 1 \Rightarrow \lim_{n \to \infty} P(n) = 0 \),

\[
P(\infty) = pr(\nu_n = \infty) = pr(E) = 1 - P \quad \text{for } n = \infty
\]

(if \( P < 1 \) the r.v. \( \nu_n \) is degenerate);

vii) aim to an estimation of the average number \( \bar{n} = \sum_{n=1}^{\infty} nP(n) \) of operations \( 0_n \) which do happen the event \( E \), if \( P = 1 \) (only in this case the r.v. \( \nu_n \) is not degenerate and the average number \( \bar{n} \) has meaning:

\[
P < 1 \Rightarrow \bar{n} = \infty
\)

viii) aim to an estimation of the probability \( P = pr(E) < 1 \) when the event \( E \) is delimited to the set of all the \( E_n \) for \( n \leq n \) by a truncation of the sequence \( \{0_n\} \) at the \( n \)-th "operation" if \( \{0_n\} \) did not lead to \( E \) for \( n \leq n \). Each \( E_n \) will be empty for \( n > n : E = E_n = \cup_{n=1}^{n} E_n \);

\[
(x_1, \ldots, x_n) \in E = E_n \iff x(n) < A(n) \quad \text{for each } n=1, \ldots, n
\]

I.2 - Such problems may be described by a binary scheme of r.w. (random walk) on a semi-infinite line \([x]\), with the a.b. (absorbing barrier) defined by a sequence \( \{A(n)\} \) \( n = 1, 2, \ldots \) as upper bound variable with the number \( n \) of steps: \( (x', x") \) is the binary system of displacements at each \( n \)-th step

\[\text{[pr}(x_n = x') = q, \ pr(x_n = x") = p, \ pr(x_0 = 0) = 1 \].
Since \( x(n) = \sum_{i=1}^{n} x_i \leq nx'' \) if \( x(m) < A(m) \) for every \( m=1,2,...,n \) we have
\[
x(n)-(n-n)x'' \leq x(m) < A(m) \Rightarrow x(n) < \min \{ A(n); A(m)+(n-m)x''; nx'' \}
\]

Then the r.w. may assume on the semi-infinite line \([x]\) the positions \( x(n) \) with

\[
(2) \quad nx' \leq x(n) \leq \bar{x}(n) = \max \{ x(n) : x(n) < \min \{ A(n); A(m)+(n-m)x''; nx'' \} \}.
\]

With regard to the sequence \( \{\bar{x}(n)\} \) we find

\[
x(n) = x(n-1)+x_n \Rightarrow \max x(n) = \bar{x}(n) = \begin{cases} 
\max x(n-1)+x''=\bar{x}(n-1)+x'' (x_n=x''), & \text{or} \\
\max x(n-1)+x'=\bar{x}(n-1)+x' (x_n=x'), & \text{or} \\
x(n-1)+x_n < \bar{x}(n-1)+x' & \text{or} \end{cases}
\]

from which

\[
(3) \quad \bar{x}(n)-\bar{x}(n-1) \leq x' \text{ or } x'' \quad (x' < x'') .
\]

Remembering that in any one sequence \( \{x(m)\} \) we have \( x(n)=x(m-1)+x_m \) for each \( m(x(0)=0) \), then if we suppose \( x(n)=x^\wedge(n)=x(n-1)+x_n^\wedge \)

\[\Rightarrow \quad E^\wedge_n \quad \text{non-empty } \quad (x_n^\wedge=x_n) \text{ necessarily we must have the reversible implications}\]

\[\exists \{x'(m)\} \text{ such that i) } x'(n-1)+x_m^\wedge=x'(m) \leq \bar{x}(m) < A(m) \text{ for } m=1,2,...,n-1 ,\]

\[\text{ii) } x'(n-1)=x(n-1) , x_n^\wedge=x_n^\wedge \iff \]

\[\iff \quad \bar{x}(n-1)+x_n^\wedge \geq x(n-1)+x_n^\wedge = x(n)=x^\wedge(n) \geq A(n) > \bar{x}(n) \]

\[\iff \quad \bar{x}(n)-\bar{x}(n-1) < x_n^\wedge .\]

Hence following (3) we arrive at this general result

\[
(4) \quad \begin{cases} 
\bar{x}(n)-\bar{x}(n-1) \leq x' \iff E^\wedge_n \quad \text{non-empty} \left( x_n^\wedge= \begin{cases} 
x'' & \text{if } \bar{x}(n)-\bar{x}(n-1)=x' \\
x' \text{ or } x'' & \text{if } \bar{x}(n)-\bar{x}(n-1) < x' \end{cases} \right) \\
\bar{x}(n)-\bar{x}(n-1) = x'' \iff E^\wedge_n \quad \text{empty} .
\end{cases}
\]

Moreover

\[
(5) \quad E^\wedge_n \text{ non-empty } \iff \bar{x}(n) < x(n)=x^\wedge(n)=x(n-1)+x_n^\wedge \leq x(n-1)+x'' \leq \bar{x}(n-1)+x'' \Rightarrow \\
\Rightarrow x^\wedge(n)+x'' \leq \bar{x}(n-1) < A(n) .
\]
The events \( E_n \) are certainly empty also for \( n < n_0 \) and \( n > n_+ \), where

\[
\begin{align*}
\begin{cases}
  n_0 &= \min(n : nx'' \geq A(n)) \\
n_+ &= \min(n : nx' \geq A(n))
\end{cases}
\] (\( n_+ \) can be inexistent: neither finite nor infinite)
\]

The integers \( n_0 \) and \( n_+ \) are respectively the lower and the upper extremity of the r.v. \( n_n \), whose range is then \( n_+ - n_0 \).

Necessary and sufficient condition for \( n_0 < \infty \), that is in order for the r.v. to be absorbed at least once after a finite number \( n \) of steps, is \( A(n) \leq nx'' \) for some finite \( n \). The non-existence of \( n_0 \) (or \( n_0 = \infty \)) is the necessary and sufficient condition for \( P=0 \).

Necessary and sufficient condition for \( n_+ > 1 \), that is in order for the r.v. to assume at least one position \( x_1 = x' \) before absorption, is \( A(1) > x' \); otherwise the scheme would be meaningless. The existence of \( n_+ \) finite or infinite, by implying the emptiness of the event \( \bar{E} \), is a sufficient condition for \( P=1 \).

But this condition is too restrictive to imply \( P=1 \). We shall give as conclusion of Chap. II a wider sufficient condition about the behavior of the sequence \( \{A(n)\} \) as \( n \to \infty \) in order for \( E \) to imply \( P=pr(E)=1 \): under that condition \( pr(\bar{E})=0 \) even if \( \bar{E} \) is not empty (\( n_+ \) inexistent).

In general from the above-mentioned problems one did not require the distribution of the r.v. number \( n_n \) of steps leading either to absorption (event \( E \)) or to no absorption (event \( \bar{E} \)), from which it would be possible draw directly and exactly the probability \( P=pr(E) \), as well as the other pertinent indexes (\( \bar{n}=E(n_n), Var(n_n), \ldots \)) if \( P=1 \). In fact a method for an approximate or direct calculation of \( \bar{E} = pr(E_n) \) has not been until now investigated in view of the complexities of solutions which besides are not considered absolutely necessary to find \( P=pr(E) \) or an its approximation. This because
generally simpler schemes have been, until now, considered based on linearly variable a.b.'s: \( A(n) = S_n + A \) with \( S \geq x', A > 0 \).

In such a case we shall show (Chap. II) that is indeed possible to apply known synthetical methods which give good estimation of the statistical features of the considered linear scheme of r.w. (P, and eventually \( \bar{n} \) if \( P=1 \)), according to the Wald's scheme for the Sequential Analysis of a Binomial Distribution.

If however the a.b. \( \{A(n)\} \) is not linear the just now mentioned synthetical methods can be applied to obtain

i) an estimation of \( P \) (if \( P < 1 \)) and \( \bar{n} \) (if \( P=1 \)) by bounding \( \{A(n)\} \) above and below by linear a.b.'s;

ii) an extensive sufficient condition about the behavior of the sequence \( \{A(n)\} \) as \( n \to \infty \) in order for the absorption probability to be \( P=1 \).

However mostly such estimations of \( P \) (if \( P < 1 \)) and \( \bar{n} \) (if \( P=1 \)) are too coarse to be utilizable. What is left is to find a direct way yielding an expression, a method of calculation of the probability \( P(n) = \text{pr}(E_n) \) for each \( n \).

Then the defective estimations \( P' = \sum_{n=1}^{n'} P(n) \) of \( P \) (if \( P < 1 \)) and \( \bar{n}' = \sum_{n=1}^{n'} nP(n) + (n'+1)(1-P') \) of \( \bar{n} \) (if \( P=1 \)) will be more precise if a larger number \( n' \) of probabilities \( P(n) \) (\( n=1,2,\ldots,n' \)) are calculated.

The description of such a method is the specific object of the present work in which (Chap. III) we shall investigate specially the case of nonincreasing or nondecreasing sequences \( \{a(n)\} = \{\frac{A(n)-nx'}{x''-x'}\} \).

The extension of the r.w. on a semi-infinite line bounded only above by an a.b. to the case of a.b.'s not linearly variable may satisfy better some problems which are solved only approximately from the schemes of r.w.
with linear a,b. At the same time it may lead to the formulation of new more general problems which would be an extension of some known ones.

1.3 - Let us indicate by "R.W. \([(-1,1);(q,p); -b < \sum_{i=1}^{n} z_i < a]\)" the classical scheme of unidimensional r.w. on the z-axis with two a,b.'s defined by \((-b)\) and \((a)\) \((ab > 1, a > 0, b > 0)\). Indicating the displacement at the n.th step by the r.v. \(z_n\), only two values are possible: \(z_n = -1\) with probability \(a=1-p\), and \(z_n=1\) with probability \(p\) independent of \(n\); \((-b)\) and \((a)\) are respectively the lower and upper boundaries of the open interval on which the r.w. is walking starting from the position \(z=0\). The possible positions of the r.w., before crossing with the a,b. and ending with the absorption by the a,b. are the integers

\[
\sum_{i=1}^{n} z_i \text{ with } -b < \sum_{i=1}^{n} z_i < a, \quad n=1,2 \ldots .
\]

Since in the following speaking of r.w. we will mean an unidimensional binomial r.w. (only two displacements necessary and incompatible at each step) with only the upper boundary as a,b., and since we will agree to attribute always the probability \(p\) to the greater value (the second one) of the two displacements, we can write for convenience

\[
\text{R.W. } \left[ (-1,1);(q,p); -\infty < \sum_{i=1}^{n} z_i < a \right] \equiv \text{R.W. } \left[ (-1,1); \sum_{i=1}^{n} z_i < a \right].
\]

The general extension of such a scheme is given by a R.W. \([(x',x'');x(n) < A(n)]\) whose two displacements of probability q and p are at each n.th step respectively \(x_n=x', x_n=x''\) with \(x' < x''\), and whose possible positions are the values \(x(n) = \sum_{i=1}^{n} x_i < A(n)\), where the sequence \(\{A(n)\}\) is an upper variable a,b.

We then observe that if from a system of displacements \((x',x'')\) we pass
to the system \((z', z'') = (ux' + v, ux'' + v)\) with invariant probabilities \((q, p)\),
this is equivalent to make the following linear transformation of the r.v.
\(X_n \rightarrow X_n' = uX_n + v\) for each \(n\), and this implies \(z(n) = ux(n) + vn\).

Since we want to maintain the inequality \(x'' > x'\), we must have \(u > 0\)
as can be seen from the relation

\[
(7) \quad x' < x'', \quad z' = ux' + v < z'' = ux'' + v \quad \iff \quad u = \frac{z'' - z'}{x'' - x'} > 0, \quad v = \frac{z'' - x'z'}{x'' - x'}
\]

If \([C(n)]\) is the a.b. of the R.W. \([(z', z''); x(n) < C(n)]\), this r.w.
scheme is equivalent to the preceding R.W. \([(x', x''); x(n) < A(n)]\) if
\(C(n) = uA(n) + vn\); in fact we have the relation

\[
(8, i) \quad x(n) = \frac{z(n) - vn}{v} < A(n) \iff z(n) = ux(n) + vn < uA(n) + vn = C(n).
\]

When \(x(n) > A(n)\) for the first time, necessarily \(z(n) > C(n)\) for the
first time:

\[
(8, ii) \quad x(n) = x^\wedge(n) > A(n) \iff z(n) = z^\wedge(n) = ux^\wedge(n) + vn > C(n)
\]

So we obtain the following relation of equivalence among r.w. schemes:

R.W. \([(x', x''); x(n) < A(n)] \equiv R.W. \([(z', z''); z(n) < C(n)]\),
where, from (7) and (8, i)

\[
(9) \quad \begin{cases}
\text{i) } C(n) = uA(n) + vn \implies \begin{cases}
(z' = ux' + v
\end{cases} \\
\text{given } (x', x''), A(n)
\end{cases}
\]

\[
\begin{cases}
\text{ii) } (z', z'') \implies \begin{cases}
z(n) = \frac{z'' - z'}{x'' - x'} x(n) + \frac{x'' z' - x' z''}{x'' - x'} n
\end{cases} \\
C(n) = \frac{z'' - z'}{x'' - x'} A(n) \frac{x'' z' - x' z''}{x'' - x'} n
\end{cases}
\]

\[
\text{iii) } A(1) > x' \implies uA(1) > ux' > z' - v \iff C(1) = uA(1) + v > ux' + v = z'
\]
(in order for the r.w. to exist).
From (4) we obtain another relation of equivalence. In fact if we consider the class \([A^0/\{A(n)\}]\) of the a.b.'s \(\{A^0(n)\}\) such that

\[
A^0(n) \in [A^0/\{A(n)\}] \implies \begin{cases} 
\text{a) } x(n) < A^0(n) \leq \bar{x}(n)+(x''-x') \\
\text{that } \bar{x}(n) \leq \bar{x}(n-1)+x' (\iff E_n \text{ non-empty}) \\
\text{b) } \bar{x}(n) < A^0(n) \text{ for every } n \text{ such that } \\
\bar{x}(n) = x(n-1)+x'' (\iff E_n \text{ empty}),
\end{cases}
\]

the sequence of events \(E_n\) does not change when we consider any a.b. \(A^0(n)\) of the class \([A^0/\{A(n)\}]\). Then this class is an equivalent class with regard to the equivalence relation which establishes the same sequence \(E_n\) (the class includes the a.b. \(A(n)\) which is its representative; the class has no lower extreme). Hence

\[
R.W. [(x',x''); x(n) < A(n)] = R.W. [(x',x''); x(n) < A^0(n)] \text{ for every } \ A^0(n) \in [A^0/\{A(n)\}].
\]

The analysis of every R.W. \([(x',x''); x(n) < A(n)]\) will be carried out more conveniently by considering the equivalent scheme R.W.\([(0,1); f(n) < a(n)]\)

where, from (9,ii) \((z'=0, z''=1, C(n)=a(n), z(n)=f(n))\), we have

\[
\begin{align*}
\text{i) } a(n) &= \frac{A(n)-nx'}{x''-x'} , \\
\text{ii) } f(n) &= \frac{x(n)-nx'}{x''-x'} \geq 0 \implies x(n) = (n-f(n)) x' + f(n)x'' , \\
\text{iii) } \bar{x}(n) &= (n-\bar{f}(n)) x' + \bar{f}(n)x''.
\end{align*}
\]

From (11,ii) we deduce \(f(n)\) is the frequency of the displacements equal to \(x''\) after \(n\) steps; that is

\[
\sum_{i=1}^{n} f_i , \text{ where } f_i = \frac{x_i-x'}{x''-x'} = \begin{cases} 
\text{f'=0 with probability q=1-p} \\
\text{f''=1 with probability p}
\end{cases}
\]

\[
f(n)-f(n-1) = f_n = 0,1 .
\]

So (2), (4), (5), (6) and (10) give
\begin{align*}
\text{i)} & \quad 0 \leq f(n) \leq \overline{f}(n) = \max \{f(n) : f(n) < \min \{a(n) : a(m+n-m; n) \} ; m \leq n \} \\
\text{ii)} & \quad n_0 = \min (n : a(n) \leq n) , \quad n_+ = \min (n : a(n) \leq 0) ; \\
\text{(13)} & \quad \begin{cases} 
\overline{f}(n) \leq \overline{f}(n-1) \iff E_n \text{ non-empty} \iff \overline{f}(n) < a^*(n) \leq \overline{f}(n) + 1, \\
\text{iii)} & \quad ((a^*(n)) \in [a^*/\{a(n)\}]); \\
\overline{f}(n) = \overline{f}(n-1)+1 \iff E_n \text{ empty} \iff \overline{f}(n) < a^*(n), \\
\text{iv)} & \quad E_n \text{ non-empty} \implies f^*(n)-1 \leq \overline{f}(n-1) < a^*(n-1). 
\end{cases}
\end{align*}
II.1 - In this Chapter we shall examine principally the general scheme

\[ R_{N}([x',x''];x(n) < A(n)] = R_{N}((0,1);f(n) < a(n)) \]

when the upper a.b. is linearly variable (especially for \( 0 < s < 1 \)):

\[ A(n) \equiv [Sn+A] \iff (a(n)) \equiv (sn+h) \text{ with } s = \frac{S-x'}{x''-x'}, \]

\[ h = \frac{A}{x''-x'} > -s \] \((h < -s \iff n_{+} = 1)\).

It would be more exact to speak of "a.b. variable according to the arithmetic progression \( (a(n)) = (sn+h) \)" because we are interested in the values of the linear function \( st+h \) for \( t=1,2,... \). However in the following we shall keep the notation of "a.b. linearly variable \( a(t) = (a(n)) \)" with the restriction just now indicated \( t=n=1,2,... \).

Then in the case where \( a(n) = sn+h \) \((h > -s)\) \((13)\) becomes

\[
\begin{align*}
(14) \quad & \quad \begin{cases}
\text{i)} & 0 \leq f(n) \leq \overline{f}(n) \leq \max [f(n) : f(n) < ns+h] ; \\
\text{ii)} & n \circ = \min (n : n \geq \frac{h}{s+1}) \iff \frac{h}{s+1} \leq n \circ < \frac{h}{s+1} + 1 , \\
& s < 0 \iff n_{+} = \min (n : n \geq \frac{-h}{s}) \iff \frac{-h}{s} \leq n_{+} < \frac{-h}{s} + 1 ; \\
& (a) s \leq 0 \text{ and } (14,1) \Rightarrow \overline{f}(n) \leq \overline{f}(n-1) \iff E_{n} \text{ non-empty, } n \leq n_{+} ; \\
\text{iii)} & b) 0 < s < 1 \text{ and } (14,1) \Rightarrow \begin{cases}
\text{either 1)} & \overline{f}(n) = \overline{f}(n-1) \iff E_{n} \text{ non-empty} , \\
& \text{or 2)} \overline{f}(n) = \overline{f}(n-1) + 1 \iff E_{n} \text{ empty} ; \\
& c) 1 \leq s \text{ and } \begin{cases}
\text{s+h > 1 } \Rightarrow \overline{f}(n) = n \iff E_{n} \text{ empty for every } n \geq 1 , \\
\text{0 < s+h \leq 1 } \Rightarrow \overline{f}(n) = n \iff E_{n} \text{ empty for every } n > 1 ; \\
& \text{iv) } E_{n} \text{ non-empty } \Rightarrow f^{\circ}(n)-1 \leq \overline{f}(n-1) < s(n-1)+h \Rightarrow \\
& \quad \Rightarrow sn+h \leq f^{\circ}(n) < sn+h+1 - s .
\end{cases}
\end{cases}
\end{align*}
\]

Since for any integer \( f \) such that \( sn+h \leq f < sn+h+1-s \) \((s < 1)\) we have \( f-1 < s(n-1)+h \), and since from \((14,iv)\) there exists always \( f(n-1) \leq \overline{f}(n-1) \) such that \( f^{\circ}(n)-1=f(n-1) \) \(( < s(n-1)+h)\) we obtain
II.2 - Now let us investigate the problem of the absorption probability

\[ s \leq n < s + h + 1 - s \implies f'(n) = f \quad \implies f'(n) \] takes all the integer values \( f \) belonging to the interval \( I(s) = [sn+h, sn+h+1-s[ \), with \( s < 1 \).

Especially we have the following assertion:

"If (and only if) \( 0 \leq s < 1 \implies f'(n) + 1 \not\in I(s) \) then \( f'(n) \), when is defined, take only an integer".

Let us consider \( s = \frac{d-1}{d} \) with \( d \) = integer > 0 \( (0 \leq s < \frac{d-1}{d} < 1) \).

Remarkably, that \( \frac{d-1}{d} n \cdot c < \frac{d-1}{d} n \) if (and only if) \( c = \frac{d-1}{d} (n' - n) \implies n' - n = t \implies c = t(d-1) \) then if \( f'(n) \) is defined for \( n = m \) we have

\[
\frac{d-1}{d} (m + t d) + h \leq f'(m) + t(d-1) \quad \text{integer} < \frac{d-1}{d} (m + t d) + h + 1 - s \quad \text{for} \quad t = 0, 1, 2, \ldots \text{ and } m + t d \geq n_0.
\]

Since by definition [(6), (14,ii)] there exists \( f'(n) = \min \), if we put \( m = n_0 \) and \( t = \frac{n_0 - m}{d} = r \) we have

\[
\frac{d-1}{d} (n_0 + r d) + h \leq n_0 + r(d-1) < \frac{d-1}{d} (n_0 + r d) + h + 1 - s \implies f'(n_0 + r d) = n_0 + r(d-1).
\]

So we obtain

\[
(16) \quad s = \frac{d-1}{d} (d \text{ integer } > 0) \implies f'(n_r) = n_r - r, \text{ for } n_r = n_0 + r d \text{ and } r = 0, 1, 2, \ldots
\]

where \( n_0 = \min \) (integer \( m : m \geq d h \)),

and this implies that the linear a,b, \( \frac{d-1}{d} n + h \) is equivalent to the a,b.

\[
\left\{ \frac{d-1}{d} n + \frac{n_0}{d} \right\}.
\]

II.2 - Now let us investigate the problem of the absorption probability

(17) \quad \text{If } s < 0 \text{ from (14,ii)} \text{ we have } \frac{h}{-s} < n_+ < \frac{h}{-s} + 1; \text{ hence } P = 1.

If \( s = 0 \) the distribution of the r.v. \( n_0 \) is Pascal's distribution and the distribution of the r.v. \( n_0 - n \) (\( n_0 = \min \) (integer \( \geq h \)) is the negative binomial distribution : as known.
If \( s \geq 1 \), with \( a(l)=s+h > l \), the barrier \((sn+h)\) is not absorbing because \( sn+h > n^* = n \). Therefore \( P = 0 \).

If \( s \geq 1 \), with \( s+h \leq l \) (obviously \( s+h > 0 \)), we have \( n+1 = 1 \), that is we have absorption only when \( f'(1)=1 \) and therefore \( P=P_1=P \).

For the cases in which \( 0 < s < 1 \), \( h > 0 \), our r.w. scheme is equivalent to Wald's scheme of a Sequential Plan for a Binomial Distribution with an indcision zone of the frequencies \( f(n) \) on the Cartesian plane \([s,f]\) included between the two parallel straight lines

\[
L_0 \equiv (f=sn+h) \quad \text{and} \quad L \equiv (f=sn+h) \quad \text{with } h_0 < 0 < h, \text{ when } h_0 \to -\infty
\]

In the following we put \( P=p_o \) if \( p < s \), \( P=p_1 \) if \( p > s \).

Now let us make the following ratios constant

\[
\frac{p_1 q_1}{sn+h \ n-sn-h} \quad \frac{p_1 q_1}{sn+h_0 \ n-sn-h_0} = A, \quad \frac{p_1 q_1}{sn+h \ n-sn-h} \quad \frac{p_1 q_1}{sn+h_0 \ n-sn-h_0} = B
\]

from which we obtain the following relations

\[
(18) \quad 0 < p_o < s < p_1 < 1, \quad h_0 < 0 < h : \quad \begin{cases} \quad A = \left( \frac{p_1}{p_o} \right)^s \left( \frac{q_1}{q_o} \right)^{1-s} n \left( \frac{p_1 q_0}{p_o q_1} \right)^h \quad \text{for each } n \\ \quad B = \left( \frac{p_1}{p_o} \right)^s \left( \frac{q_1}{q_o} \right)^{1-s} n \left( \frac{p_1 q_0}{p_o q_1} \right)^h \quad \text{for each } n \end{cases}
\]

\[
\Rightarrow \left( \frac{p_1}{p_o} \right)^s \left( \frac{q_1}{q_o} \right)^{1-s} = 1 \Rightarrow \frac{p_1 q_0}{p_o q_1} = \left( \frac{q_0}{q_1} \right)^1 > 1 \Rightarrow
\]

\[
\Rightarrow 0 \leq B = \left( \frac{p_1 q_0}{p_o q_1} \right)^h = \left( \frac{q_0}{q_1} \right)^h < 1. \quad \text{A} = \left( \frac{p_1 q_0}{p_o q_1} \right)^h = \left( \frac{q_0}{q_1} \right)^h
\]

The equation \( \left( \frac{X}{p} \right)^s \left( \frac{1-X}{1-p} \right)^{1-s} = 1 \) for \( 0 < s < 1 \) gives always two real and different solution (coinciding if \( p = s \)) \( p_o \) and \( p_1 \) with \( p_o < s < p_1 \);
or \( p=p_1 > s \), hence if \( p \) is given as either \( p=p_0 < s \) we are interested in the other solution \( g(p,s) \). But there is not an algebraic expression giving directly \( g(p,s) = \begin{cases} 
 p_1 & \text{if } p = p_0 \\
 p_0 & \text{if } p = p_1 
\end{cases} \) as a function of \( p \) and \( s \). However in practice, in every concrete case, we can get numerical approximation to the solution \( g(p,s) \).

Let us then assume that \( g(p,s) \) is practically known; we will get immediately the values of \( A=A(h) \) and \( B=B(h_0) \). The probability that for some finite \( n \) would be for the first time \( f(n) < \text{sn}+h \) (from (14,iv) \( f^*(n) < \text{sn}+(h+1-s) \), with \( \text{sm}+h_0 < f(m) < \text{sm}+h \) for each \( m < n \), from [1] is

\[
\begin{align*}
\alpha : & \quad \frac{1-\beta(h_0)}{A(h+1-s)} < \alpha = \alpha(h_0) \leq \frac{1-\beta(h_0)}{A(h)} \quad \text{when } p = p_0 \\
\beta : & \quad \beta = \beta(h_0) \leq B(h_0) \quad \text{when } p = p_1
\end{align*}
\]

(19)

The straight line \( L_0 = \text{sn}+h_0 (h_0 < 0) \) is a lower bound put in the initial R.W. \([(0,1), f(n) < \text{sn}+h] \) which can be considered as limit of the scheme R.W. \([(0,1), \text{sn}+h_0 < f(n) < \text{sn}+h] \) when \( h_0 \to -\infty \).

Then, because \( \frac{p_1 q_0}{p_0 q_1} > 1 \) and from this \( h \to -\infty \) \( B(h) = \lim_{h \to -\infty} \frac{p_1 q_0}{p_0 q_1} h_0 \to 0 \Rightarrow \), we have

\[
\begin{align*}
\left(1 + \frac{1}{A(h+1-s)}\right) \leq P = \lim_{h_0 \to -\infty} \alpha(h_0) \leq \frac{1}{A(h)} \quad \text{if } p = p_0 < s ,
\end{align*}
\]

(20)

\[
\begin{align*}
P = \lim_{h_0 \to -\infty} (1-\beta(h_0)) \quad \text{if } p = p_1 > s
\end{align*}
\]

that is

\[
\begin{align*}
\left(\frac{p_0 q_1}{p_1 q_0}\right)^{h+1-s} \leq P \leq \left(\frac{p_0 q_1}{p_1 q_0}\right)^{h} \quad \text{if } p = p_0 < s ,
\end{align*}
\]

(21)

\[
\begin{align*}
P = 1 \quad \text{if } p = p_1 > s
\end{align*}
\]

If \( p_0 = p_1 = s \) then \( \frac{p_0 q_1}{p_1 q_0} = 1 \) and we have \( P = 1 \).

Recapitulating all cases examined about the couple \( (s,p) \) with
\(-\infty < s < +\infty , 0 \leq p \leq 1\), we can write

\[
P(s) = \begin{cases} 
  1 & \text{for } p > s \ (s < 1), \\
  \frac{p_0 q_1}{p_1 q_0}^{s+1-s} P(s) \leq \left(\frac{p_0 q_1}{p_1 q_0}\right)^{s} & \text{for } p = p_0 < s < p_1 < 1, h \geq 0, \\
  0 & \text{for } s \geq 1.
\end{cases}
\]

If we consider \( s = \frac{d-r}{d} \) \((d = \text{integer} > 0)\) from (16) we have exactly

\[
f(n) = \frac{d-1}{d} n + \frac{n_0}{d} \text{ for } n=n_r=n_o+rd \ (r=0,1,2,...). \]

Then (20) gives exactly

\[
P = \frac{1}{n^2}.
\]

Hence we have

\[
P = \begin{cases} 
  p_0 < s = \frac{d-1}{d} => P(s) = P \ (d-1) \left(\frac{p_0 q_1}{p_1 q_0}\right)^{n_0} \frac{d}{n}.
\end{cases}
\]

For instance, if \( d=2 \) we have \( s = \frac{1}{2} => p_o = 1-p_1 => \left(\frac{p_0 q_1}{p_1 q_0}\right)^{n_0} = \left(\frac{p_0}{p_1}\right)^{n_0} < \frac{1}{2} \)

and therefore \( P = \left(\frac{p}{q}\right)^{n_0}. \) We remark that

\[
\text{R.W. } \left[(0,1);f(n) < \frac{1}{2} n + \frac{n_0}{2}\right] = \text{R.W. } \left[(-1,1),z(n) < n_o\right]
\]
is the scheme usually used for the classical problem of the gambler's ruin who having \( n_o \) is playing, against an infinitely rich adversary, a series of games in each of which \( p \) is the constant probability of losing the stake 1.

II.3 - In all cases in which \( P=1 \) \((p=p_1 \geq s)\) we have the problem of the average \( \bar{n} = n(s,p) \) of the number \( n \) of steps required for absorption. We can again utilize the scheme of Wald's Sequential Analysis employing the Average Sample Number function of the Sequential Probability Ratio Test.

After putting \( \frac{p_0 q_1}{p_0 q} = \left(\frac{q_0}{q}\right)^{\frac{1}{s}} = e^u \) where from (13)

\[
p = p_1 \Rightarrow \left(\frac{q_0}{q_1}\right)^{\frac{1}{s}} > 1 \Rightarrow u = \frac{1}{s} \log = \frac{q_0}{q_1} > 0 \ (s > 0), \]

let us consider the equivalence

\[
\text{R.W. } \left[(0,1);f(n) < sn+th\right] \equiv \text{R.W. } \left[(z',z'';z(n) < hu\right].
\]
From (9,i), where \( A(n) = sn+h, C(n) = hu \), we have

\[
u(sn+h)+vn=mu \Rightarrow us+v=0 \Rightarrow v=-su = 1g \frac{a}{q_o} \text{ from which } (x'=0, x''=1)
\]

\[
\begin{cases}
  z' = ux'+v = 1g \frac{a}{q_o} = -us \\
  z'' = ux''+v = 1g \frac{pq_o}{p_o q} + 1g \frac{a}{q_o} = 1g \frac{p}{p_o} = u(1-s) \text{ with probability } p;
\end{cases}
\]

Therefore

\[
(25) \quad \text{R.W. } [(0,1); f(n) < sn+h] = R.W. \left[ 1g \frac{a}{q_o}, 1g \frac{p}{p_o}; z(n) < \frac{h}{s} 1g \frac{q_o}{q} \right].
\]

Besides we have from (8,ii) and (24,i)

\[
z^\wedge(n) = f^\wedge(n)(z''-z') + nz' > hu \Rightarrow \frac{z^\wedge(n)}{u} = f^\wedge(n) - ns,
\]

and after from (14,iv)

\[
h < \frac{z^\wedge(n)}{u} = f^\wedge(n) -ns < h+1-s \Rightarrow hu < z^\wedge(n) < (h+1-s)u \Rightarrow
\]

\[
\Rightarrow hu < E(nz^\wedge(n)) < (h+1-s)u.
\]

Remembering the Wald's equation [1]

\[
(26) \quad E(nz^\wedge(n)) = E(z^\wedge(n)) E(z^\wedge(n))
\]

we obtain the following interval of estimation of the average number

\[
\bar{n} = E(nz^\wedge(n)) = \bar{E}(s,p):
\]

\[
(27) \quad s < p : \frac{mu}{E(nz^n)} \leq n(s,p) < \frac{(h+1-s)u}{E(nz^n)}
\]

where \( E(nz^n) = z'^p + z''q = 1g \left( \frac{p}{p_o} \right)^p \left( \frac{a}{q_o} \right)^q \), \( u=1g \frac{pq_o}{p_o q} \).

But if \( s=p \) we obtain \( \bar{n} = \infty \). In fact \( s=p \Rightarrow p_o = p \Rightarrow \lim_{s \to p} \frac{mu}{E(nz^n)} =
\]

\[
= h \lim_{p_o \to p} \frac{u}{E(nz^n)} = \frac{0}{0},
\]

from which we have, following l'Hopital's rule,
\[ \bar{n}(p,p) = \frac{h}{p} \lim_{p_o \to \infty} \frac{d \nu}{dp_o} \frac{dp_o}{dE(nz_n)} = \frac{h}{p} \lim_{p_o \to p} \frac{1}{pq} \frac{1}{1-p-o - p} = \]

\[ = \frac{h}{pq} \lim_{\epsilon \to 0} \frac{1}{\epsilon} = \infty \quad (\frac{h}{pq} > 0) \]

that is

(28) \[ \bar{n}(p,p) = \bar{n}(s,s) = \infty \]

For \( s = \frac{d-1}{d} \) (d integer \( > 1 \)) when \( p > \frac{d-1}{d} \) and \( n_o = \text{min} \) (integer \( \geq dh \))
we have exactly

(29) \[ \bar{n}(\frac{d-1}{d},p) = \frac{n_o}{E(nz_n)d} \log \frac{pq_o}{p^o q} . \]

For \( s=0 \) (\( d=1 \): Pascal's distribution) we have exactly

(30) \[ \bar{n}(0,p) = \frac{n_o}{p} . \]

However for \( s < 0 \), except the obvious and useless inequality

\[ \frac{h}{1-s} < \bar{n}(s,p) < -\frac{h}{s} \quad (\frac{h}{1-s} \text{ and } -\frac{h}{s} \text{ are the the abscissae of the points of contact of the straight line } f=sn+h \text{ with the straight lines } f=n \text{ and } f=0) \]

there are no synthetic methods able to give a good estimation of \( \bar{n}(s,p) \).

In the following (III-2.) we will see that it is possible to use in this case \( (s < 0) \) a direct analytic method which gives the exact distribution of \( n \) from which we obtain the exact value of \( \bar{n}(s,p) \).

II-4. If now we consider a scheme R.W. \([0,1]; f(n) < a(n)\] where \( \{a(n)\} \)
is an whatever a,b. not linear function of \( n \), in some cases the results obtained above may be used with advantage according to the validity of the approximation which will be now discussed.
We remark that the probability \( P \) of absorption by the a.b. \( \{a(n)\} \) as well as the average number \( n(\text{if } P=1) \) is a function of the sequence 

\[ \{a(n)\} : P=P(\{a(n)\}) \]

Moreover from (10) and (13) we have

\[ P(\{a(n)\}) = P(\{a^*(n)\}) \quad \text{for each a.b. } \{a^*(n)\} \in \{a^*//\{a(n)\}\}. \]

Given two a.b.'s \( \{a'(n)\} \) and \( \{a''(n)\} \), remembering the definition of the events \( E \) and \( \bar{E} \) (1.1) and putting \( P'=P(E')=P(\{a'(n)\}) \), \( P''=P(E'')=P(\{a''(n)\}) \), \( \bar{P}(E')=1-P' \), \( P(\bar{E}'')=1-P'' \), we have \( P'-P''=P(E'')-P(E') \).

If there exist \( \{a^*(n)\} \in \{a^*//\{a'(n)\}\} \) and \( \{a''(n)\} \in \{a''//\{a''(n)\}\} \) such that \( a^*(n) \leq a''(n) \) for each \( n \geq n_0=\min\{n:a^*(n) \leq n\} \) then for every sequence \( \{f_n\} \rightarrow \{f(n)\} \) such that \( f(n) < a^*(n) \) for each \( n \), we have also \( f(n) < a''(n) \) for each \( n \); but the converse generally is not true: hence

\[ a^*(n) < a''(n) \iff (f_n \in \bar{E}' \iff f_n \in \bar{E}'') \iff \bar{E}' \subseteq \bar{E}'', \]

from which \( P' P''=P(E''-\bar{E}') \geq 0 \). This result is summarized in the following

**THEOREM 1.** If there exist two a.b.'s \( \{a^*(n)\} \) and \( \{a''(n)\} \) equivalent to the given a.b.'s \( \{a'(n)\} \) and \( \{a''(n)\} \) respectively, such that \( a^*(n) \leq a''(n) \) for each \( n \geq n_0 \), then \( P'=P(\{a'(n)\}) \geq P''=P(\{a''(n)\}) \).

**COROLLARY 1.** i) [for each \( n, \exists n \in \mathbb{N} > n \) such that \( a^*(m) \leq a^*(n) \) ==> 

\[ \bar{E}' \subseteq \bar{E}' \Rightarrow P'=P'' ; \]

ii) \( P''=1 \Rightarrow P'=1 \).

So, given the a.b. \( \{a(n)\} \), if it is possible to find an equivalent a.b. \( \{a^*(n)\} \) and two straight lines \( a'(n)=s'n+h' \) and \( a''(n)=s''n+h'' \) such that \( s'n+h' \leq a^*(n) \leq s''n+h'' \) for each \( n \geq n_0 \), then we have

\[ P' \geq P \geq P'', \quad \text{where } P=P(\{a(n)\}), P'=P(\{s'n+h'\}), P''=P(\{s''n+h''\}), \]

with \( P' \) and \( P'' \) available from the above considered methods (II.2, 22).
Hence Corollary 1.ii) and (22) give us the following

**Theorem 2.** Given the a.b. \( \{a(n)\} \), if there exists an equivalent a.b. \( \{a^*(n)\} \) and a straight line \( \{s''+h'\} \) such that \( s''+h'' \geq a(n) \) \( (n \geq n_o) \), and if \( p > s'' \), then we have: \( p \geq s'' \Rightarrow P'' \Rightarrow P '' = P \ ((s''+h'') = l \Rightarrow P = 1 \).

Putting \( \overline{E} = \overline{E}_a \) when the a.b. is \( \{a(n)\} \), let

\[ b(n) = \min [a(m) : m > n] \Leftrightarrow \{b(n)\} = \min \{(a'(n)) : a'(n) \leq a(n), \overline{E}_a = \overline{E}_a \} ; \]

the sequence \( \{b(n)\} \) is nondecreasing. Then from Corollary 1.i) we have

\[ \overline{E}_b \Leftrightarrow \overline{E}_a \Rightarrow P = P ((a(n))) = P ((b(a))) = P ((b^*(n))) \] for each \( \{b^*(n)\} \in [b^*/\{b(n)\}] \)

Now let \( \{b(n)\} \) be the lower extremum of the class \( [b^*/\{b(n)\}] \):

i) R.W. \( ((0,1); f(n) < b^*(n)) \Leftrightarrow 0 \leq f(n) \leq \overline{f}(n) = \overline{b}(n) ; \)

ii) \( \{b(n)\} = L \{a(n)\} \), where \( L \) is the law of the correspondence \( \{a(n)\} \rightarrow \{b(n)\} \)

From (10) we know \( \{b(n)\} \) does not belong to the class \( [b^*/\{b(n)\}] \).

In this way the greatest inferior bound for \( P \) is

\[ \max \{\overline{P}'' : P'' \leq P\} = \max \{P ((s''+h'')) : s''+h'' > \overline{b}(n)\} \leq P. \]

By definition \( \{b(n)\} \) is nondecreasing sequence, with \( \overline{b}(n+h)-\overline{b}(n) = 0,1 \) and \( \{n-\overline{b}(n)\} \) nondecreasing: hence the upper limit of the sequence \( \{\overline{b}(n)/n\} \) is always defined, and we have

\[ 0 \leq \overline{s} = \lim_{n \rightarrow \infty} \frac{\overline{b}(n)}{n} \leq 1. \]

Moreover by definition of \( \{a^*(n)\}, \{b^*(n)\} \) and \( \{\overline{b}(n)\} \) we have

\[ \lim_{n \rightarrow \infty} \frac{b^*(n)}{n} \geq \overline{s}, \quad \lim_{n \rightarrow \infty} \frac{a^*(n)}{n} \geq \overline{s} \] for each \( \{b^*(n)\} \in [b^*/\{b(n)\}] \) and

\[ \{a^*(n)\} \in [a/\{a(n)\}]. \]

Therefore

\[ \overline{s} = \min \{\overline{s}'' : s''+h'' > \overline{b}(n), h'' < \infty \}. \]

Having found \( h_o'' \) such that \( \overline{s}n+h_o'' > \overline{b}(n) \) there exists always \( \{b^*(n)\} \in [b^*/\{b(n)\}] \) such that \( \overline{s}n+h_o'' > b^*(n) > \overline{b}(n) \) for each \( n(\geq n_o) \).

Since \( P ((b^*(n))) = P ((b(n))) = P ((a(n))) = P \) we have from (32)

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\[ P \geq P'' = P([sn+h'']) \]. So Theorem 2 becomes the following

**THEOREM 3.** Given the a.b. \{a(n)\}, if \( p \geq \bar{s} = \lim_{n \to \infty} \frac{\overline{b}(n)}{n} \), where \([b(n)] = I[a(n)]\), then \( P = P([a(n)]) = 1 \).

Given two a.b.'s \{a'(n)\} and \{a''(n)\}, let \( P' = P([a'(n)]) = \sum_{n=1}^{\infty} P'(n) \), \( P'' = P([a''(n)]) = \sum_{n=1}^{\infty} P''(n) \), with \( P'(n) \) \( P''(n) \) = probability of absorption after \( n \) steps by the a.b. \{a'(n)\}.

Under the conditions of Theorem 1 the conditional probability \( P''(n|m) \) is defined, that for some sequence \{f(i)\} there occurs \( f(n) \geq f''(n) \) when we have \( f(m) = f'(m) \geq f''(m) \) (we remember that in the same sequence \{f(i)\}, \( f'(n) \) is the first value for which \( f(n) \geq a(n) \).

Obviously \( P''(n|m) = 0 \) for \( m > n \), \( m < n' \), and \( n < n'' \). So we have

\[ P''(n) = \sum_{m=n''}^{n} P''(n|m)P'(m) \], \( \sum_{n=m}^{\infty} P''(n|m) = 1 \),

and we can consider the conditional expected value \( \overline{n''}(m) = \sum_{m=n''}^{\infty} n P''(n|m) \geq m \).

Hence if \( P' = P'' = 1 \), putting \( \overline{n'} = \overline{n}([a'(n)]) \) and \( \overline{n''} = \overline{n}([a''(n)]) \), we have

\[ \overline{n''} = \sum_{n=n''}^{\infty} n P''(n) = \sum_{n=n''}^{n'} n P''(n) = \sum_{m=n'}^{\infty} P'(m) \sum_{n=m}^{\infty} n P''(n|m) = \sum_{m=n'}^{\infty} \overline{n''}(m)P'(m) \geq \sum_{m=n'}^{\infty} m P'(m) = \sum_{n=n'}^{\infty} n P'(n) = \overline{n'} \),

from which we obtain the following

**THEOREM 4.** If there exist two a.b.'s \{a''(n)\} and \{a''''(n)\} equivalent to the given a.b.'s \{a'(n)\} and \{a''(n)\} respectively such that \( a''(n) \leq a''''(n) \) for each \( n \geq n'' \) and if \( P' = P'' = 1 \), then \( \overline{n'} \leq \overline{n''} \).
Under the conditions which give us (32), if \( p \geq s''(\geq s') \) from (22) and (32) we have \( p = p' = p'' = 1 \). Hence, remembering (27), we obtain the following

**THEOREM 5.** Given the a.b. \( \{a(n)\} \), if it is possible to find an equivalent a.b. \( \{a^*(n)\} \in[a^*/a(n)] \), and two straight line \( \{s'n+h'\} \) and \( \{s''+h''\} \) such that

1. \( s'n+h' \leq a^*(n) \leq s''+h'' \) for each \( n \geq n_o \),
2. \( p \geq s' (\geq s'' \geq 0) \), then we have

\[
\bar{n}' = \frac{h'u}{E(n'z_n)} \leq \bar{n} \leq \frac{(h''+1-s'')u}{E(n''z_n)} = \bar{n}'' .
\]
CHAPTER III (Non-linear Absorbing Barrier)

III.1. Let us observe that when in R.W. \([0,1); f(n) < a(n)\] the a.b. \([a(n)]\) is not linear, its trend may be too far from a linear one: in this case in (32) and (33) we shall have values \(s', s'', h', h''\) implying differences \(P'-P'' > 0\) and \(\bar{n}'' - \bar{n}' = (h'' + 1 - s'' - h') \frac{n}{\mathbb{P}(\bar{n}z_n)} > 0\) which can be too wide for a more precise estimation of \(P\), if \(P < 1\), and of \(\bar{n}\), if \(P = 1\).

The following cases are the ones in which the same problem occurs even when the a.b. is linear \([sn+h]\):

i) \(s < 0\) and we ask for \(\bar{n}\) (\(P = 1\))

ii) \(s > 0\), but a truncation of the r.w. takes place when after \(\bar{n}\) steps it has not yet reached the a.b. \([ns+h]\) (the two events \(E(n)\) "absorption before the first \(n\) steps", and \(E(n)\) "truncation at the \(n\).th step" are necessary and incompatible).

Such an a.b. is not linear; it coincides with the arithmetical progression \([sn+h]\) only for the first \(n\) terms.

Due to the truncation, the probability of absorption will be

\[
\sum_{i=1}^{n} P(i) = P/\bar{n} < P; \text{ where } P \text{ is the known probability of absorption when no truncation occurs (} n = \infty), \text{ and the difference } P - P/\bar{n} \text{ increases as } n \text{ decreases, so that, if } \bar{n} \text{ is not sufficiently large, an evaluation of } P/\bar{n} \text{ by } P \text{ may lead to coarse errors.}
\]

However that may be, in order to analyse non-linear schemes it is more convenient (or necessary, depending on the situation) to apply other analytical methods which give the probabilities \(P(n)\).

Then from the sequence \([P(n)]\) we can further calculate directly \(P (if P < 1)\) or the average number \(\bar{n}\) and other characteristic features.
(if P=1). Moreover to settle directly whether P=1 we can use Theorem 3 from Chapter II.

Generally, following (13, iv), the event $E_n$ can be realized in one of $k(n)$ ways according to the values $f_i^{\uparrow}(n)=f_i(n)+i-1$ ($i=1, 2, \ldots, k(n)$) (if $E_n$ is empty we put $k(n)=0$ and there is no value $f_i^{\uparrow}(n)=f(n-1)+1$).

Let $E_{n;i}$ be the single event corresponding to $f_i^{\uparrow}(n)$ and $C(n;i)$ the number of arrangements of $n$ displacements in which there are $f_i^{\uparrow}(n)$ displacements equal to $f'' (=1)$ without obtaining any $f^{'m}(m)$ for each $m < n$; then we have

$$C(n;i) \leq \binom{n}{f_i^{\uparrow}(n)}; \quad P(n;i)=\Pr(E_{n;i})=C(n;i)p^{f_i^{\downarrow}(n)}q^{n-f_i^{\downarrow}(n)}.$$  

Therefore the calculation of $P(n) = \sum_{i=1}^{k(n)} P(n;i)$ (the $k(n)$ events $E_{n;i}$ are incompatible) can be reduced to a practical method for the calculation of the coefficients $C(n;i)$.

III-2. - The one-dimensional R.W. $[(0,1); f(n) < a(n)]$ corresponds to a particular case of two-dimensional r.w. in the semi-infinite first quadrant of the Cartesian plane $[t,f](t \geq 0, f \geq 0)$ bounded by an absorbing line $(a,b.)$ whose function is $f=a(t) \geq 0$ for $t > 0$ such that

$$a(n) = \lim_{t \to n} a(t) \text{ for each } n=1,2,\ldots .$$

The initial position of the two-dimensional r.w. is at the origin $O=(0,0)$, and its system of displacements at each $n$th step is represented by the random vector $\vec{f}_{n} = \vec{f}_{n} = (1, f(n))$ which take the vector value $\vec{f}_{n} = \vec{f}' = (1, f') = (1, 0)$ with probability $q=1-p$ and the vector value $\vec{f}_{n} = \vec{f}'' = (1, f'') = (1, 1)$ with probability $p$: that is $\Pr(\vec{f}_{n} = (1,f)) = p \frac{1-f}{q}$.  

Let $\vec{f}(n) = \sum_{i=1}^{n} \vec{f}_{i}$. It follows that the two-dimensional r.w. can take only the positions defined by the vectors or points $\vec{f}(t) = (t,f(t))$, ...
with integer coordinates: \( t=n=1,2,\ldots \); \( f(n)=0,1,2,\ldots \), \( \overline{t}(n) \) (see (13,i).

At each step one can go only from a point or vector \( \overline{t}(n)=(n,f(n)) \) to the point \( \overline{t}(n+1)=(n+1,f(n+1)) \), where \( f(n+1)=f(n)+f(n) \Rightarrow \overline{t}(n+1)=\overline{t}(n)+\overline{t}(n) \).

Therefore each position \( \overline{t}(n) \) can be reached if and only if the number of steps is \( n \). Moreover a necessary and sufficient condition for considering the sequence \( \{f(i)\} \rightarrow \{f(i)=(i,f(i))\} \) (\( i=1,2,\ldots,n \)) as the result of a sequence \( \{0_i\} \) of \( n \) steps or trials \( 0_i \rightarrow f_i \rightarrow f(i) \) is that \( f(i+1)-f(i)=f=0,1 \) for every \( i=1,2,\ldots,n \): we shall call such a sequence "sequential" with regard to the system of displacements \((f',f'')=(0,1)\).

Any particular arrangement of the two displacements \( f' \rightarrow f' \) and \( f'' \rightarrow f'' \) which are repeated at random in the same sequence \( \{f(n)\} \rightarrow \{f(n)\} \), can be graphically represented by drawing segments between successive points of the sequential sequence \( \{\overline{t}(n)\} \equiv \{(n,f(n))\} \).

We obtain then a non-decreasing graph with horizontal unit segments (parallel to the \( n \)-axis) and upward unit segments (parallel to the straight line \( f=n \)) which correspond to the vectors \( \overline{t}_n=(1,0) \), \( \overline{t}_n=(1,1) \) respectively.

Such a graph we will call a "sequential step graph" (s.s.g.) starting from the origin \( O \): it will be said of order \( n \) if limited to \( n \) consecutive unit segments.

Given a point \( (n,a) \) (\( 0 \leq a \) = integer \( \leq n \)), the s.s.g.'s of order \( n \) which end at it \((E f_i=a)\) will be called "converging to \( (n,a) \)" , and the one which originates from \( (n,a) \) considered from the \( (n+1) \)-th step, will be called "starting from \( (n,a) \)".

The number of s.s.g.'s of order \( n \) converging to \( (n,a) \) (starting from \( O \)) is the binomial coefficient \( \binom{n}{a} \), i.e. the number of arrangements with repetition of \( n \) elements of which \( a \) are equal to \( f'' \) and \( (n-a) \) are equal to \( f' \). The probability of each of these \( \binom{n}{a} \) s.s.g.'s is

\[
\prod_{i=1}^{n} \frac{f_i-1}{f_i} = p^a q^{n-a}.
\]
Obviously \( \binom{n}{a} = 0 \iff a < 0 \text{ or } a > n \) gives the condition \( 0 \leq a = f(n) \leq n \) which is satisfied by any s.s.g.

Every s.s.g. which, starting from \((n',a')\), converges to \((n,a)\) is equivalent to a s.s.g. of order \( n-n' \); so the number of such s.s.g.'s will be \( \binom{n-n'}{a-a} \). Of course \( 0 \leq a-a' \leq n-n' \).

It follows immediately that the number of s.s.g.'s of order \( n \) which converge to \((n,a)\) and go through \((n',a')\) (i.e. converging to and starting from \((n',a')\)) is the product \( \binom{n'}{a} \binom{n-n'}{a-a} \).

Let \( \{(n_r,a_r)\} \) be a system of \( N \) points \((r=1,2,\ldots,N)\) such that for every point \((n_r,a_r)\) there exists at least one s.s.g. \( (f(n)) \) which goes through it \( f(n_r) = a_r \) without going through any other point \((n_j,a_j)\) of the system with \( n_j < n_r \) \((n_r \text{ and } a_r \text{ are integers})\). The point \((n_r,a_r)\) is the first point of the system met by the s.s.g. \( (f(n)) \).

If we limit such a s.s.g. just to the order \( n_r \), we can say such a s.s.g. is absorbed by the system in the first point \((n_r,a_r)\) to which the s.s.g. converges: the system \( \{(n_r,a_r)\} \) as well as its points, will be called "absorbing".

Then we can say that a point \((n_r,a_r)\) of the absorbing system is "following" the point \((n_j,a_j)\), which is "preceding" the point \((n_r,a_r)\), if \( n_r - n_j \geq a_r - a_j > 0 \). We will call "subordinate" to the point \((n_j,a_j)\) every "following" point \((n_r,a_r)\). For if the point \((n_j,a_j)\) absorbs, all those s.s.g.'s which go first through it, will stop at the order \( n_j \), and will not go on and converge to any absorbing point \((n_r,a_r)\) following the preceding point \((n_j,a_j)\).

If there are no other absorbing points \((n_j,a_j)\) preceding the point \((n_r,a_r)\), this non-subordinate point is said to be "primary" and \( \binom{n_r}{a_r} \) s.s.g. of order \( n_r \) converge to it. If then \((n_r,a_r)\) is not primary, the number of
s.s.g's absorbed by this point is \( C_r < \binom{n_r}{a_r} \) since some of the s.s.g.'s which would have been absorbed by \((n_r,a_r)\), if it had been primary, do not reach it since they are absorbed by the absorbing points which "precede" it.

Then, if \((n_j,a_j)\) precedes \((n_r,a_r)\), among the \( \binom{n_r}{a_r} \) s.s.g's which would converge to the point \((n_r,a_r)\) if it was primary there are instead \( C_{r,j} \) ones absorbed (and stopped at the \( n_j \) order) from the preceding point \((n_j,a_j)\); \( C_{r,j} \) is the number of s.s.g's of order \( n_r \) which converge to \((n_r,a_r)\) going through \((n_j,a_j)\).

Since \( C_j \) is the number of s.s.g's absorbed by \((n_j,a_j)\), and \( \binom{n_r-n_j}{a_r-a_j} \) the number of s.s.g's starting from \((n_j,a_j)\) and converging to \((n_r,a_r)\) without restrictions, we have

\[
C_{r,j} = C_j \binom{n_r-n_j}{a_r-a_j}.
\]

If we indicate by \( j_h, h=1,2,\ldots,t_r \), the subscripts \( j \) of the \( t_r \) points \((n_j,a_j)\) preceding \((n_r,a_r)\) and we put \( j_{t_r+1}=r \), we have the following relation

\[
\sum_{h=1}^{t_r+1} C_{r,j_h} = \sum_{h=1}^{t_r+1} C_{j_h} \binom{n_r-j_h}{a_r-a_{j_h}} = \binom{n_r}{a_r}
\]

from which we obtain the following recurrence formula for the coefficients

\[
C_r = C_{t_r+1}
\]

(34) \[
C_r = \binom{n_r}{a_r} - \sum_{h=1}^{t_r} C_{j_h} \binom{n_r-n_j}{a_r-a_{j_h}}.
\]

Since the \( C_r \) s.s.g's which are absorbed by \((n_r,a_r)\) are incompatible and each has probability \( p_r q_r \) the probability of the absorption event \((n_r,a_r)\) is

(35) \[
P_r = C_r p_r q_r^{n_r-a_r}.
\]
By definition of "absorption," the $N$ absorption events $(n_r, a_r)$ of the system $\{(n_r, a_r)\}$ are incompatible, too. Therefore the total probability of absorption by the system $\{(n_r, a_r)\}$ is

$$P = \sum_{r=1}^{N} P_i = \sum_{r=1}^{N} C_r p^r q^{N-r}.$$  

If the $N$ absorption events $(n_r, a_r)$ are also necessary we have $P=1$; in this case we say the absorbing system is "closed".

For instance, if $n_r = n$ constant for every $r$, then for the system to be closed it is necessary that $a_r = r=0,1,2,...,n$ $(N=n+1)$. The points $(n, r)$ are primary (not subordinate to each other): therefore $C_r = \binom{n}{r}$, and so $r$ has a binomial distribution.

If $N < \infty (N=\infty)$ the absorbing system is finite (infinite). We call "asymptotically closed" an absorbing system which is closed ($P=1$) and infinite $(N=\infty)$, with $a_r > 0$ for every finite $r$ and $\lim_{r \to \infty} a_r > 0$: the r.w. converges to the absorbing system almost certainly $(N=\infty, P=1,$ but a sequence $\{f(n_r)\}$ with $f(n_r) < a_r$ for every $r$ finite or infinite is a non-impossible event: $P(E) = 0$ but $E$ is not empty).

III-3. When a sequence $\{a(n)\}$ is given as a.b., such an a.b. is equivalent to the system of actually absorbing points $\{(n_t, a_t)\}$ $(t=1,2,..,N)$ located upon or beyond the a.b. $\{a(n)\}$. We have $a_t = \text{integer} \geq a(n_t)$, but the same value $n$ can correspond to $k(n)$ different subscripts $t$:

$$t_i \rightarrow n_{t_i} = n \rightarrow a_{t_i} \geq a(n) \Rightarrow a_{t_i} = f_i^*(n), \; i=1,2,...,k(n).$$

So the absorbing system $\{(n_t, a_t)\}$ this time is exactly the system $\{(n_r, f_i^*(n_r))\}$ where $i=1,2,...,k(n_r)$, $n \in [n_r]$ $\iff$ $k(n) > 0$ $(r=0,1,2,...)$, $n \notin [n_r] \iff k(n) = 0 \iff E_n$ empty (no $f_i^*(n)$ is defined for $n$) and
\[ \bar{f}(n_x) < a(n) < f^\wedge_1(n_x), \quad f^\wedge_1(n_x) < f^\wedge_{i+1}(n_x). \]

Since from (13,iii)

\[ \bar{f}(n_x) \text{ and } \bar{h} \text{ such that } \bar{f}(n_x) < f(n_x) < f^\wedge_{\bar{h}}(n_x) \Rightarrow \exists \{f(1), f(2), \ldots, f(n_x)\} \]

such that

i) \( f(m) \leq \bar{f}(m) \), \( f(m)-f(m-1)=0,1 \) for each \( m=1,2,\ldots,n_x-1 \) \( (f(0)=0) \)

ii) \( f(n_x-1) \leq f^\wedge_{\bar{h}}(n_x)-1 \leq \bar{f}(n_x-1) \),

iii) \( f(n_x)=f^\wedge_1(n_x) > \bar{f}(n_x), \quad f(n_x)-f(n_x-1)=1 \)

then the assumption \( \bar{f}(n_x) < f^\wedge_1(n_x) < f^\wedge_{i+1}(n_x) \) implies

\[ \begin{cases} 
  f^\wedge_{i+1}(n_x) = f^\wedge_i(n_x) + 1 \\
  f^\wedge_i(n_x) = \bar{f}(n_x) + 1 
\end{cases} \Rightarrow k(n_x) = f^\wedge_k(n_x) - \bar{f}(n_x). \]

Obviously, from (13,iv),

\[ f^\wedge_k(n_x) = \bar{f}(n_x-1) + 1; \]

but from (13,iii) \( E_{n_x-1} \) is empty for \( i=1,2,\ldots,n_x-n_x-1-1 \) we have

\[ \bar{f}(n_x-1)-\bar{f}(n_x-2)+1=\bar{f}(n_x-3)+2=\ldots=\bar{f}(n_x-1)+n_x-n_x-1-1. \]

Therefore

\[ f^\wedge_{k(n_x)}(n_x) = \bar{f}(n_x-1)+n_x-n_x-1, \]

and from (37)

\[ k(n_x) = n_x-n_x-1-\left(\bar{f}(n_x)-\bar{f}(n_x-1)\right). \]

If we want an a.b. \( \{a(n)\} \) such that \( k(n) \geq 1 \) \( (\Rightarrow \Rightarrow E_n \text{ non-empty}) \) for every \( n \) of the range \((n_0, n_+)^{n_n}\), we obtain from (40) in accordance with (13,iii)

\[ n_x-n_x-1 = 1 \text{ for each } r=0,1,2,\ldots \Rightarrow k(n) = \bar{f}(n)-\bar{f}(n-1)+1 \text{ for each } n \Rightarrow \]

\[ \Rightarrow \bar{f}(n)-\bar{f}(n-1) = k(n)-1 \geq 0 \Rightarrow \bar{f}(n) \text{ is a non-increasing sequence.} \]
(Conversely, if $\{f(n)\}$ is a non-increasing sequence then from (13,iii) we have $E_n$ non-empty for every $n$ such that $n_o \leq n \leq n_+$).

Let us consider then an a.b. $\{a(n)\}$ such that to it there corresponds a non-increasing sequence $\{f(n)\}$ for $n_o \leq n \leq n_+$ ($f(n') = n'$ for $n' < n_o$).

This occurs, for instance, when the a.b. is linear $\{sn+h\}$ with $s < 0$.

From (41), (38) and (37) we have

$$f_{k(n)}^\wedge(n) = f(n-1)+l = f_{l-1}^\wedge(n-1).$$

Hence every absorbing point $(n, f_{k(n)}^\wedge(n))$ is not primary and all the s.s.g.'s which converge to it pass through $(n-1, f_{k(n)}^\wedge(n)-1) = (n-1, f(n-1))$, which is a primary non-absorbing point because by hypothesis $f(m) > f(n-1)$ for each $m < n-1$. Therefore

$$C(n, k(n)) = \left(\frac{n-l}{f(n-1)}\right).$$

On the contrary, since from (37), (41) and (42) it follows that

$$f_{l}^\wedge(m) > f_{l}^\wedge(n) \text{ for each } m < n \text{ and } i < k(n)$$

every absorbing point $(n, f_{l}^\wedge(n))$ with $i < k(n)$ is primary.

Moreover from (42) if $k(m) > 1$, $k(m+1)=k(m+2)=\ldots=k(n-1)=1$, $k(n) > 1$ we have $f_{l}^\wedge(m) = f_{k(n)}^\wedge(n-1)+l$. So

$$i < k(n), 0 \leq f \leq n_{o}-1 : \quad f_{i}^\wedge(n) = f \implies n=n(f), \text{ which is a single-valued non-increasing function of } f.$$  

Therefore

$$i < k(n) : \quad C(n, i) = \left(f_{i}^\wedge(n)\right) = \left(n(f)\right) \text{ with } 0 \leq f \leq n_{o}-1.$$  

There are no other absorbing points besides the ones considered in the two systems $\{(n, f_{k(n)}^\wedge(n))\}$ (this set includes all the points with $k(n) = 1$)
and \( \{(n(f), f)\} \) with \( n_0 \leq n \leq n_+ \), \( 0 \leq f < n_0 - 1 \). Hence \( N = n_+ + 1 \). The union of the two systems gives us the absorbing system of the r.w. scheme considered.

Concluding, we have the following probabilities for the two types of absorbing points

\[
\begin{align*}
\{ P_n = \Pr((n, k(n))) \} = \left( \frac{n-1}{f(n-1)} \right) p_{f(n-1)+1} q^{n-1} r_f(n-1), & \quad n_0 \leq n \leq n_+ \\
\{ P_f = \Pr((n(f), f)) \} = \left( \frac{n(f)}{f} \right) p f n(f) r_f, & \quad 0 \leq f \leq n_0 - 1.
\end{align*}
\]

Because the sequence \( \{ f(n) \} \) is non-increasing the absorbing system is always closed (asymptotically closed if \( n_+ = \infty \) and \( \lim_{n \to \infty} f_1^*(n) > 0 \)). Hence

\[
P = \sum_{n=n_0}^{n_+} P_n + \sum_{f=0}^{n_0 - 1} P_f = 1, \quad \bar{n} = \sum_{n=n_0}^{n_+} n P_n + \sum_{f=0}^{n_0 - 1} n(f) P_f.
\]

In the case in which \( f = n_0 \) constant, it is necessarily true that \( f(n) = n_0 - 1 \), \( k(n) = 1 \) for every \( n \geq n_0 \). Therefore

\[
C(n, k(n)) = C_n = \binom{n-1}{n_0 - 1}, \quad P_n = p_0 \binom{n-1}{n_0 - 1} q^{n-n_0}
\]

and the distribution of \( n-n_0 \) is the negative binomial distribution.

Since it is known that \( P = 1 \), the infinite system of absorbing points \( \{(n, n_0)\} \) is asymptotically closed.

III-4. In practical applications in which the scheme

\[\text{R.W. } \left[ (0, 1); f(n) < a(n) \right] \]

becomes more appropriate, the case of a non-decreasing a.b. \( \{a(n)\} \) generally occurs. In this case we will give to the recurrent expression of the absorption coefficients a form which is more
convenient both for practical reasons of swifter calculation and for theoretical reasons of facilitating a more detailed analysis of such coefficients.

To a nondecreasing a,b. (a(n)) there corresponds a nondecreasing sequence \{\overline{f}(n)\}. So in the following work we shall consider the equivalent a,b. (a^*(n)) = \left[ a^*/[a(n)] \right] where \( a^*(n) = \overline{f}(n) + 1 \).

Since \((n, \overline{f}(n-1))\) is the upper s.s.g. which is not absorbed, we call the equivalent a,b. (a^*(n)) = (f(n)+1) "sequential".

From (37) and (38) we obtain

\[ k(n) = \overline{f}(n-1) - \overline{f}(n) + 1, \]

and since \( \overline{f}(n) \) is a s.s.g., from (13,iii) we have

\[ \overline{f}(n) - \overline{f}(n-1) = \begin{cases} 0 & \iff k(n) = 0 \iff n \in \{ n_r \} \iff f(n-1) = a^*(n) = f^*(n), \\ 1 & \iff k(n) = 0 \iff c(n) = 0 \ (f^*(n) \ is \ not \ defined). \end{cases} \]

So the nonincreasing absorbing system \((n_r, a_r)\) \((r=0,1,2,\ldots)\) is defined, where

\[ \begin{align*} a_r &= a^*(n_r) = f^*(n_r - 1) + 1 = f(n_r - 1) + 1, & a_0 &= n_0. \end{align*} \]

Then, since \( n_r - 1 < n_r \) and \( a_r - 1 = a_r \), (34) gives us the following recurrence formula for the coefficients \( c(n_r) \)

\[ c(n_r) = c_r = \left( \frac{n_r}{a_r} \right) - \sum_{i=0}^{r-1} c_i \left( \frac{n_r - i}{a_r - a_i} \right). \]

In order for the absorbing system \((n_r, a_r)\) to be infinite, the sequence \( (n_r - a_r) \), which is positive non-decreasing, must be divergent. A finite non-increasing absorbing system would not have practical interest, unless the event "truncation" is considered.

If we put \( n_r - n_r - 1 = d_r \geq 1 \) then (51) and (52) give us

\[ k(n_r - i) = 0 \iff \overline{f}(n_r - i) = \overline{f}(n_r) - 1 \quad \text{for} \ i = 1, 2, \ldots, d_r \]

from which
\[ D(n,a) = D(n-1,a) + D(n-1,a-1) \]

Hence from \( d_r = n_r - n_{r-1} = a_r - a_{r-1} + 1 \) we obtain by recurrence

\[ n_r - a_r = n_{r-1} - a_{r-1} + 1 = n_{r-2} - a_{r-2} + 2 = \ldots = n_0 - a_0 + r \]

and since \( a_0 = n_0 \) we find the useful relation

\[ n_r - a_r = r \quad \text{for each} \quad r=0,1,2,\ldots . \]

We note that

\[
\begin{align*}
\text{(56)} & \quad \left\{ \begin{array}{l}
\text{i) } n_r = \min (n: n-a^*(n) = r) \\
\text{ii) } a_r = \min (a^*(n): n-a^*(n) = r).
\end{array} \right.
\end{align*}
\]

Then it follows that the knowledge of the increasing sequence of integers \( \{n_r\} \) is sufficient to represent a "sequential" a. b. \( \{a^*(n)\} \).

Letting \( C(n_r) = c_r \), with the new assumption (53) becomes

\[
\begin{align*}
\text{(57)} \quad c_r &= \binom{n_r}{n_r - r} - \sum_{i=0}^{r-1} c_i \binom{n_r - n_i}{n_r - r - n_i + 1} = \binom{n_r}{r} - \sum_{i=0}^{r-1} c_i \binom{n_r - n_i}{r - i},
\end{align*}
\]

and the probability \( P(n_r) = p_r \) of absorption by \( (n_r, n_r - r) \) becomes

\[
\text{(58)} \quad p_r = \text{pr} \left( n_r = n_r \right) = c_r p r q ^ {n_r - r} .
\]

Letting \( D(n,a) \) indicate the number of s.s.g.'s \( \{f(1), \ldots , f(n) = a\} \) which converge to the non-absorbing point \( (n,a) \) with \( a < a^*(n) = \bar{f}(n)+1 \) and \( f(m) \leq \bar{f}(n) \) for each \( m < n-1 \), we have

\[
\text{(59)} \quad D(n,a) = D(n-1,a) + D(n-1,a-1)
\]

because if \( f(n) = a \) then necessarily \( f(n-1) = a-1, a \).

We put \( C(n,a) = D(n-1,a-1) \). Since every s.s.g. \( \{f(1), \ldots , f(n_r - 1), \ldots , f^*(n_r) = a_r \} \) which reaches the absorbing point \( (n_r,a_r) \)
necessarily passes through the points \((n_r-i, f(n_r-i)) \Rightarrow f(n_r-i) = f(n_r-i)\) with \(i=1, s, \ldots, d_r\), we have

\[
C(n_r, a_r) = C(n_r-i+1, a_r-i+1) = D(n_r-1, a_r-1) \quad \text{for } i=1, 2, \ldots, d_r.
\]

So by (58, ii) and (59) we obtain

\[
i=d_r \Rightarrow C(n_r, a_r) = D(n_r-1, a_r-1) = D(n_r-1, a_r-1) +
\]

\[
+ D(n_r-1, a_r-2) = C(n_r-1, a_r-1) + C(n_r-1, a_r-1),
\]

that is, remembering \(C(n_r, a_r) = C(n_r) = C_r\)

(60) \(C_r = C_{r-1} + C(n_r-1, a_r-1)\).

This result will be useful later.

III-5. After choosing an integer \(r\), let us make a first change in the sequential a.b. \((a^\wedge(n))\) by making the following change (dependent on \(r\)) in the sequence \(\{n_i\} \rightarrow \{a^\wedge(n_i) = n_{-r}\}\)

\[
(n_0, n_1, \ldots, n_s, n_s+1, n_s+2, \ldots, n_r, n_{r+1}, \ldots)
\]

where

i) \(s=s(r)\) such that \(n_s < n_0 + r < n_s+1 \leq n_{s+1}\);

ii) \(n_{s+j} = n_{o+r+j}\) for each \(j=1, 2, \ldots, r-s\);

iii) \(a'_{s+j} = n_{s+j}-(s+j) = n_{o+r-s}\) constant for each \(j=1, 2, \ldots, r-s\);

We put \(C'_{s+j} = C(n'_s, a'_{s+j})\)

For the following developments we also introduce

(61, i) \(n'_s = n_0 + r, \quad n'_s-s = a(n_0 + r) = n_0 + r-s\) \((j=0)\\

(to which there will correspond the auxiliary coefficient \(C'_s\) such that

(61, ii) \(C'_s = \begin{cases} C_s & \text{if } n_s = n'_s \\ 0 & \text{if } n_s < n'_s \end{cases}\)

\(n'_s = n_0 + r\)\).
Applying (53) we have

\begin{equation}
C'_r = \binom{n}{r} - \sum_{h=0}^{s} \binom{n'-n}{r-h} C_h - \sum_{h=s+1}^{r-1} \binom{n'-n}{r-h} C'_h .
\end{equation}

We now consider the difference

\begin{equation}
C'_r - C'_{r-1} = \binom{n'}{r} - \binom{n'-1}{r-1} - \sum_{h=0}^{s} \left[ \binom{n'-n}{r-h} - \binom{n'-1-n}{r-1-h} \right] C_h - \sum_{h=s+1}^{r-2} \left[ \binom{n'-n}{r-h} - \binom{n'-1-n}{r-1-h} \right] C'_h - \binom{n'-1-n'}{r-1-h} C'_h + \binom{n'-n'}{r-1} \cdot C'_{r-1} .
\end{equation}

Since, by definition, \( n'_r - n'_h = r-h \), \( n'_{r-1} - n'_h = r-1-h \), \( n'_{r-1} = n'_r \), on applying the known relation

\begin{equation}
\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b} \quad (a \geq b)
\end{equation}

we obtain

\begin{align}
i) \quad \left( \frac{n'}{r-h} \right) - \left( \frac{n'-1}{r-1-h} \right) = & \begin{cases} 
\binom{n'-1-n}{r-h} & \text{for } m=n_r, \ h \leq s \\
0 & \text{for } s < h \leq r-2, \ m=n'_h \\
1 & \text{for } h = r-1, \ m=n'_h 
\end{cases} \\
ii) \quad \left( \frac{n'}{r} \right) - \left( \frac{n'-1}{r-1} \right) = \binom{n'-1}{r} .
\end{align}

So (63) becomes

\begin{equation}
C'_r - C'_{r-1} = \binom{n'}{r} - \binom{n'-1}{r-1} - \sum_{h=0}^{s} \binom{n'-1-n}{r-h} C_h - C'_{r-1} .
\end{equation}

We then have for \( C'_r \) a new expression as a function of the first \( s+1 \) values of \( C_h (h=0,1,2,...,s) \)

\begin{equation}
C'_r = \binom{n'-1}{r} - \sum_{h=0}^{s} \binom{n'-1-n}{r-h} C_h .
\end{equation}

We now calculate the finite differences of \( C'_r \) of order 1.st, 2.nd, ..., (r-s).th, applying (64) directly;
\[ \Delta c_r^{i} = c_r^{i-1} - c_r^{i} = \binom{n_r^{i-1}}{r} - \binom{n_r^{i-1}}{r-1} - \sum_{h=0}^{s} \binom{n_r^{i-1-h}}{r-h} - \binom{n_r^{i-1-h}}{r-1-h} \]
\[ c_r^{i} = \binom{n_r^{i-2}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-2-h}}{r-h} c_h, \]
\[ \Delta c_r^{i-1} = c_r^{i} - c_r^{i-1} = \binom{n_r^{i-2}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-2-h}}{r-h} = \binom{n_r^{i-3}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-3-h}}{r-h} c_h. \]
\[ \Delta^{(2)} c_r^{i} = \Delta c_r^{i-1} = 3c_r^{i} - 3c_r^{i-1} = \binom{n_r^{i-3}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-3-h}}{r-h} c_h, \]
\[ \Delta^{(2)} c_r^{i-1} = c_r^{i-2} + c_r^{i-3} = \binom{n_r^{i-4}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-4-h}}{r-h} c_h. \]
\[ \Delta^{(3)} c_r^{i} = \Delta^{(2)} c_r^{i-1} = 3c_r^{i} - 3c_r^{i-2} - c_r^{i-3} = \binom{n_r^{i-4}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-4-h}}{r-h} c_h. \]

\[ \Delta^{(k)} c_r^{i} = \sum_{i=0}^{k} (-1)^i \binom{k}{i} c_r^{i-1} = \binom{n_r^{i-k-1}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-k-1-h}}{r-h} c_h. \]

\[ \Delta^{(r-s)} c_r^{i} = \sum_{i=0}^{r-s} (-1)^i \binom{r-s}{i} c_r^{i-1} = \binom{n_r^{i-r+s-1}}{r} - \sum_{h=0}^{s} \binom{n_r^{i-r+s-1-h}}{r-h} c_h. \]

Since \( n_r^{i-r+s}=n_0+r \) and \( n_h-n_0-s(n_h) > a(n_0)-1=n_0-1 \) for each \( h=0,1,\ldots,s \), we have

\[ (n_r^{i-r+s-1-h})-(r-h)=n_0-1-a(n_0)=r-s \Rightarrow \binom{n_r^{i-r+s-1}}{r-h}=0, \]

from which follows the relation

\[ \sum_{i=0}^{r-s} (-1)^i \binom{r-s}{i} c_r^{i-1} = \binom{n_r^{i-r+s-1}}{r} = \binom{n_0+r-1}{r} (s=s(r), r=0,1,2,\ldots). \]
This gives us a new expression for $C_r^i$ as a function of the $(r-s)$ preceding coefficients $C_{r-1}^i$ $(i=1,2,...,r-s)$ (for the definition of $C_s^i$ see (61,ii))

\[
C_r^i = \binom{n+1}{r} - \sum_{i=0}^{r-s} (-1)^i \binom{r-s}{i} C_{r-i}^i
\]

Putting $i=r-s-j \rightarrow j=0,1,...,r-s$ we also have

\[
C_r^i = \binom{n+1}{r} - \sum_{j=0}^{r-s-l} (-1)^{r-s-j} \binom{r-s}{j} C_{s+j}^i
\]

Let us now make a second change in the sequence $\{n_n\}$ so as to have

\[
(n_0, n_1, \ldots, n_{s+I_1}, \frac{n''_{s+I_1+1}}{n_{s+I_1+1}}, \ldots, n_r, n_{r+1}, \ldots)
\]

where

i) $I_1$ is such that $n_{s+j} = n_{s+j}$ for $j=1,2,...,I_1 \leq r-s-1$

ii) $n''_{s+j} = n_{o+r+j+1}$

iii) $\alpha''_{s+j} = n''_{s+j} - (s+j) = n_{o+r+s+1}$

We put $C''_{s+j} = C(n''_{s+j}, \alpha''_{s+j})$ and also introduce $n''_{s+I_1} = n_{s+I_1+1} \rightarrow C''_{s+I_1} = 0$

Since $n''_{s+j} = n_{s+j+1}, n''_{s+j-1} = n_{s+j-1}$ for each $j=I_1+1, I_1+2,...,r-s$

we obtain $C''_{s+i} = C(n''_{s+i}, \alpha''_{s+i}) = C(n''_{s+i-1}, n''_{s+i-1} - (s+i-1)) = C(n''_{s+i-1}, \alpha''_{s+i-1} - (s+i-1))$. So applying (60) to the sequence (70) we have

\[
C(n''_{s+j}, \alpha''_{s+j}) = C(n''_{s+j-1}, \alpha''_{s+j-1}) + C(n''_{s+j-1}, \alpha''_{s+j-1})
\]

where

$C(n''_{s+j}, \alpha''_{s+j}) = C_{s+j}, \quad C(n''_{s+j-1}, \alpha''_{s+j-1}) = C_{s+j-1}$

Therefore

\[
C''_{s+j} - C''_{s+j-1} = C''_{s+j}
\]
Then applying (71) to the expression for \( C'_r \) in (69) we obtain

\[
C'_r = C'_r = C''_s+(r-s) = \sum_{j=0}^{I_1} (-1)^{r-s-j} \binom{r-s}{j} C_{s+j} + \sum_{j=I_1+1}^{r-s-1} (-1)^{r-s-j} \binom{r-s}{j} \left( C''_{s+j} - C''_{s+j-1} \right) + \sum_{j=I_1+1}^{r-s-2} \left[ (-1)^{r-s-j} \binom{r-s}{j} \sum_{j=0}^{I_1} (-1)^{r-s-j} \binom{r-s}{j} C_{s+j} \right] + \sum_{j=I_1+1}^{r-s-1} (-1)^{r-s-(r-s-1)} \binom{r-s}{r-s-1} C''_{s+(r-s-1)}.
\]

Remembering \( C''_{s+I_1} = 0 \), \( (r-s)^+ \binom{r-s}{j} = (r-s+1)^+ \binom{r-s}{j+1} \), \( (-1)^{r-s-j} = (-1)^{r-s-j+1} \)

and transferring \( C''_{s+(r-s-1)} \) to the last member of (72) we obtain

\[
C''_r = C''_s+(r-s) = \binom{n_0+r-1}{n_0-1} \sum_{j=0}^{I_1} (-1)^{r-s-j} \binom{r-s}{j} C_{s+j} - \sum_{j=I_1+1}^{r-s-1} (-1)^{r-s-j} \binom{r-s}{j} C_{s+j} + \sum_{j=I_1+1}^{r-s-1} (-1)^{r-s-(r-s-1)} \binom{r-s}{r-s-1} C''_{s+(r-s-1)}.
\]

Let us again make a change (the third one) in the sequence \( \{n_i\} \) so as to have

\[\{n_0, n_1, \ldots, n_{s+I_2}, n'''_{s+I_2+1}, n'''_{s+I_2+2}, \ldots, n'''_{r}, n_{r+1}, \ldots\}\]

where

i) \( I_2 \) is such that \( n'''_{s+j} = n_{s+j} \) for \( j=I_1+1, I_1+2, \ldots, I_2 \leq r-s-1 \)

ii) \( n'''_{s+j} = n_0 + r + j + 2 \) for \( j=I_2+1, I_2+2, \ldots, r-s \).

iii) \( a'''_{s+j} = n'''_{s+j} - (s+j) = n_0 + r + s + 2 \)

We put \( C'''_{s+j} = C(n'''_{s+j}, a'''_{s+j}) \) and also introduce

\[n'''_{s+I_2} = n_{s+I_2} + 2 \rightarrow C'''_{s+I_2} = 0.\]

By proceeding in the same way as in the previous case of \( I_1 \), we find
analogously the relation $C_{s+j}^{''} - C_{s+j-1}^{''} = C_{s+j}^{''}$ from which follows

$$
\begin{align*}
C_r^{''} &= \binom{n_o + r - 1}{r} - \sum_{j=0}^{I_1} (r-s)^j C_{s+j} - \sum_{j=I_1+1}^{I_2} (r-s+1)^j C_{s+j} - \sum_{j=I_2+1}^{r-s-1} (r-s+2)^j C_{s+j}. \\
\end{align*}
$$

Continuing in this way:

A) we make a sequence of changes in the sequence $\{n_i\}$ so as to have

$$
\{n_0, n_1, n_2, \ldots, n_s + I_k, n_{s+I_k + 1}, n_{s+I_k + 2}, \ldots, n_r, \ldots\}
$$

in conformity with each component of the non-decreasing sequence of integers

$$
\{I_0, I_1, I_2, \ldots, I_k, \ldots, I_d\} \quad \text{(for } k=1, 2, 3, \text{ we have used the superfixes 'i', 'ii', 'iii')} \text{.)}
$$

where

1. $I_k$ is such that $n_{s+j} = n_{s+j}$ for $j=I_{k-1} + 1, I_{k-1} + 2, \ldots, I_k \leq r-s-1$,

2. $n_{s+j} = n_{s+j} + 1 = n_{s+j} + k = n_o + r + j + k$  \quad for $j=I_{k-1} + 1, I_{k-1} + 2, \ldots, r-s$,

3. $a_{s+j}^{(k+1)} = n_{s+j} + (s+j) = n_o + r + s + k$

with

$$
I_0 = -1, I_d = r-s-1,
$$

$$
\text{where } d = a^\ast(n_o) - a^\ast(n_o + r) = n_o - 2r - n_o + s \quad (a^\ast(n_o + r) = n_o + r - s);
$$

B) we put $C_{s+j}^{(k+1)} = C\left(n_{s+j}^{(k+1)}, a_{s+j}^{(k+1)}\right)$;

C) we introduce $C_{s+I_k}^{(k+1)} = n_{s+I_k + k} \rightarrow C_{s+j}^{(k+1)} = 0$;

D) we use each time the recurrence relation

$$
C_{s+j}^{(k+1)} - C_{s+j-1}^{(k+1)} = C_{s+j}^{(k)};
$$

we arrive to find the following recurrence result for $k=0, 1, 2, \ldots, d$

$$
\begin{align*}
C_r^{(k+1)} &= \binom{n_o + r - 1}{r} - \sum_{i=0}^{I_{i+1}} \sum_{j=I_{i+1}}^{I_{i+1}} (-1)^{r-s-j}(r-s+i)^j C_{s+j} - \sum_{j=I_{d+1}}^{I_d} (-1)^{r-s-j}(r-s+k)^j C_{s+j}. \\
\end{align*}
$$
When finally we consider \( k = d \) we have returned to the initial a.b. \((a(n)) \to ((n, n - r))\). In fact \( I_d \) is such that \( n^{(d)}_{s+j} = n_{s+j} \) for

\[ j = I_{d-1} + 1, I_{d-1} + 2, \ldots, I_d = r - s - 1 \]

and \( n^{(d+1)}_{s+j} = n_{o+r+j+d} = n_{r+s+j} \) for

\[ j = I_d + 1 = r - s \] only; hence \( n^{(d+1)}_{s+j} = n_r \), too. Then \( c^{(d+1)}_r = c_r \).

Therefore we have

\[
(76) \quad c_r = \binom{n_0 + r - 1}{r} - \sum_{k=0}^{d-1} \sum_{j=I_k+1}^{I_{k+1}} (-1)^{r-s-j} \binom{r-s+k}{j+k} c_{s+j} = \\
= \binom{n_0 + r - 1}{n_0 - 1} - \sum_{k=0}^{d-1} \sum_{j=I_k+1}^{I_{k+1}} (-1)^{r-s-j} \binom{r-s+k}{r-s-j} c_{s+j}.
\]

Since \( n_{s+j} = n^{(k+1)}_{s+j} = n_{o+r+j+k} \) for \( j = I_k + 1, I_k + 2, \ldots, I_{k+1} \), we find

\[
(77) \quad k = k(j) = n_{s+j} - n_0 - r - j
\]

which shows that \( k = k(j) \) is a single-valued non-decreasing function of \( j = 0, 1, 2, \ldots, r - s - 1 \). In fact \( k(j) = k \) (constant) for each \( j = I_k + 1, \ldots, I_{k+1} \) and

\[ k(I_{k+1}) - k(I_k) = n_{s+I_k + 1} - n_{s+I_k} - 1 \geq 0. \]

Then by inverting the subscripts of summation \((k, j)\) in \((76)\) where \( j = 0, 1, 2, \ldots, r - s - 1 \), and \( k = k(j) \), we obtain, finally, the final recurrence formula giving the number \( c_r \)

\[
(78) \quad c_r = \binom{n_0 + r - 1}{n_0 - 1} - \sum_{j=0}^{r-s-1} (-1)^{r-s-j} \binom{n_{s+j} - n_0 - s-j}{r-s-j} c_{s+j}.
\]

Putting \( j = r - s - 1, i = 1, 2, \ldots, r - s \), we have also
This formula not only gives us $C_r$ as a function of the $r-s-1$ preceding absorbing coefficients $C_{r-i}$, but also gives us directly a simple way of constructing an automatic method of recurrent calculation which may be used profitably even when an electronic computer is not available.

In fact we can write $C_r = h_{r,0} \binom{n_0+r-1}{n_0-1} - \sum_{i=1}^{r-s} (-1)^i h_{r,i} C_{r-i}$ with

\[ h_{r,i} = \binom{n_r-i-n_0-r+1}{i} \]

and consider the sequence of coefficients $C_r$ decomposed into their components. Starting from any $r$ we find that the terms $h_{r,0}, h_{r+1,1}, h_{r+2,2}, \ldots, h_{r+i,i}, \ldots, h_s, r-s$ are the first $r-s+1$ binomial coefficients in the development of the Newton's binomial to the power $(n_r-r-n_0)$; and such terms repeat themselves for all $r' > r$ such that $n_r-r'=a(n_r)=n_r-r$.

**Remark** - All the problems and results shown in this work remain unchanged when a binary r.w. with a lower a.b. is considered.

In this case we can apply the equivalence

$$\text{R.W.}[(0,1);(q,p);f(n) < a(n)] = \text{R.W.}[(0,1);(q',p');f(n) > b(n)]$$

where $q'=p$, $p'=q$, $g(n) = n-f(n)$ (= frequency of the displacements equal to $f'=0$ in $n$ steps), $b(n) = n-a(n)$. Moreover

$g(n) = n-f(n)$, $g'(n) = n-f'(n) \leq b(n)$, $b_r = n_r-a_r = r$, etc.
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<td>14</td>
<td>14</td>
<td>26</td>
<td>12</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
\mathbf{r} & \quad \mathbf{n}_{r-r-n_0} \\
0 & \quad 0 & \begin{pmatrix} 0 \end{pmatrix} \\
1 & \quad 0 & \begin{pmatrix} 0 \end{pmatrix} \\
2 & \quad 0 & \begin{pmatrix} 0 \end{pmatrix} \\
3 & \quad 1 & \begin{pmatrix} 1 \end{pmatrix} \\
4 & \quad 1 & \begin{pmatrix} 1 \end{pmatrix} \\
5 & \quad 3 & \begin{pmatrix} 3 \end{pmatrix} \\
6 & \quad 6 & \begin{pmatrix} 6 \end{pmatrix} \\
7 & \quad 8 & \begin{pmatrix} 8 \end{pmatrix} \\
8 & \quad 8 & \begin{pmatrix} 8 \end{pmatrix} \\
9 & \quad 9 & \begin{pmatrix} 9 \end{pmatrix} \\
10 & \quad 10 & \begin{pmatrix} 10 \end{pmatrix} \\
11 & \quad 10 & \begin{pmatrix} 10 \end{pmatrix} \\
12 & \quad 10 & \begin{pmatrix} 10 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
r_{r+1} & \quad \begin{pmatrix} 0 \end{pmatrix} \\
r_{r+2} & \quad \begin{pmatrix} 1 \end{pmatrix} \\
r_{r+3} & \quad \begin{pmatrix} 2 \end{pmatrix} \\
r_{r+4} & \quad \begin{pmatrix} 3 \end{pmatrix} \\
r_{r+5} & \quad \begin{pmatrix} 4 \end{pmatrix} \\
r_{n_0} & \quad \begin{pmatrix} 5 \end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
i=0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad \" a \quad \" \\
\end{align*}
\]
<table>
<thead>
<tr>
<th>r</th>
<th>( c_r = h_{r,0} \left( \binom{n_0 + r - 1}{n_0 - 1} \right) - \sum_{i=1}^{r-s} (-1)^i h_{r,i} c_{r-1} )</th>
<th>( p_r = c_r p^a q^r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( c_0 = \left( \frac{3}{2} \right) )</td>
<td>( p_0 = c_0 p^4 )</td>
</tr>
<tr>
<td>1</td>
<td>( c_1 = \left( \frac{4}{2} \right) )</td>
<td>( p_1 = c_1 p^4 q )</td>
</tr>
<tr>
<td>2</td>
<td>( c_2 = \left( \frac{5}{2} \right) )</td>
<td>( p_2 = c_2 p^4 q^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( c_3 = \left( \frac{6}{2} \right) )</td>
<td>( p_3 = c_3 p^5 q^3 )</td>
</tr>
<tr>
<td>4</td>
<td>( c_4 = \left( \frac{7}{2} \right) + \left( \frac{1}{1} \right)c_3 )</td>
<td>( p_4 = c_4 p^5 q^4 )</td>
</tr>
<tr>
<td>5</td>
<td>( c_5 = \left( \frac{8}{2} \right) + \left( \frac{1}{1} \right)c_4 )</td>
<td>( p_5 = c_5 p^7 q^5 )</td>
</tr>
<tr>
<td>6</td>
<td>( c_6 = \left( \frac{9}{2} \right) + \left( \frac{3}{1} \right)c_5 )</td>
<td>( p_6 = c_6 p^{10} q^6 )</td>
</tr>
<tr>
<td>7</td>
<td>( c_7 = \left( \frac{10}{3} \right) + \left( \frac{6}{1} \right)c_6 - \left( \frac{3}{2} \right)c_5 )</td>
<td>( p_7 = c_7 p^{12} q^7 )</td>
</tr>
<tr>
<td>8</td>
<td>( c_8 = \left( \frac{11}{3} \right) + \left( \frac{8}{1} \right)c_7 - \left( \frac{6}{2} \right)c_6 + \left( \frac{3}{1} \right)c_5 )</td>
<td>( p_8 = c_8 p^{12} q^8 )</td>
</tr>
<tr>
<td>9</td>
<td>( c_9 = \left( \frac{12}{3} \right) + \left( \frac{8}{1} \right)c_8 - \left( \frac{8}{2} \right)c_7 + \left( \frac{6}{3} \right)c_5 )</td>
<td>( p_9 = c_9 p^{13} q^9 )</td>
</tr>
<tr>
<td>10</td>
<td>( c_{10} = \left( \frac{13}{3} \right) + \left( \frac{9}{1} \right)c_9 - \left( \frac{8}{2} \right)c_8 + \left( \frac{8}{3} \right)c_7 - \left( \frac{6}{4} \right)c_5 )</td>
<td>( p_{10} = c_{10} p^{14} q^{10} )</td>
</tr>
<tr>
<td>11</td>
<td>( c_{11} = \left( \frac{14}{3} \right) + \left( \frac{10}{1} \right)c_{10} - \left( \frac{9}{2} \right)c_9 + \left( \frac{8}{3} \right)c_8 - \left( \frac{8}{4} \right)c_7 + \left( \frac{6}{5} \right)c_6 )</td>
<td>( p_{11} = c_{11} p^{14} q^{11} )</td>
</tr>
<tr>
<td>12</td>
<td>( c_{12} = \left( \frac{15}{3} \right) + \left( \frac{10}{1} \right)c_{11} - \left( \frac{10}{2} \right)c_{10} + \left( \frac{9}{3} \right)c_9 - \left( \frac{8}{4} \right)c_8 + \left( \frac{8}{5} \right)c_7 - \left( \frac{6}{6} \right)c_6 )</td>
<td>( p_{12} = c_{12} p^{14} q^{12} )</td>
</tr>
</tbody>
</table>
BIBLIOGRAPHY


[2] FELLER, W., An Introduction to Probability-Theory and Its Applications,