ON MAXIMAL AND MINIMAL SUB-FIELDS OF CERTAIN TYPES

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Let $(X, A, P)$ be a given statistical model. We concern ourselves with particular families of sub-fields of $A$, and, for each such family we ask ourselves whether the family has maximal and minimal elements with respect to the natural partial order of inclusion relation. For example the family $(S)$ of sub-fields that are sufficient for $A$ has received a great deal of attention from theoretical statisticians. Clearly, $A(S)$ and is the maximum element of $(S)$. It is known [4] that, in general, $(S)$ has no minimal element. However, if we assume that $P$ is dominated by a $\sigma$-finite measure then it can be shown [1] that $(S)$ has an essentially minimum element $S_0$, i.e., given any $S_1 \in (S)$ and any $A \in S_0$, there exists $B \in S_1$ such that

$$P(A \Delta B) = 0 \text{ for all } P \in \mathcal{P},$$

where $\Delta$ stands for the operation of symmetric difference.

Let $B$ be a fixed sub-field of $A$ and let $(C)$ be the family of all sub-fields that are independent of $B$, i.e., for any such $C$ it is true that

$$P(BC) = P(B)P(C) \text{ for all } B \in \mathcal{B}, C \in \mathcal{C}, \text{ and } P \in \mathcal{P}.$$ 

The minimum element of $(C)$ is the trivial sub-field consisting of the null-set and the whole space. We have the following

**Theorem 1:** For any sub-field $C$ that is independent of $B$, there exists at least one maximal subfield $C^*$ such that $C \subset C^*$.

The proof of Theorem 1 is given in the next section.

For the next problem, let us suppose that the family $\mathcal{P}$ is indexed by
two parameters $\theta$ and $\varphi$, i.e.,

$$P = \{P_{\theta,\varphi}\}, (\theta, \varphi) \in X \Phi.$$ 

Consider all sub-fields $\{D\}$ that are generated by statistics whose probability distributions do not involve the parameter $\varphi$, i.e., each $D$ is a sub-field such that, for every $D \in D$, $P_{\theta,\varphi}(D)$ is a function of $\theta$ only.

The trivial sub-field is again the minimum element of $\{D\}$; but, does this family have maximal elements? We have

Theorem 2: Given any sub-field $D$ such that the restriction of $P_{\theta,\varphi}$ to $D$ does not involve $\varphi$, there exists a maximal such sub-field $D^*$ that contains $D$.

The above Theorem is an immediate generalization of Theorem 1 in [2].

In the next section we give the proof of the above two theorems. In the final section we comment on some further problems of the same kind.

2. Proofs of Theorems 1 and 2

We need the following well-known lemmas.

**Lemma 1 (Zorn's Lemma):** If for a partially ordered set it is true that every linearly ordered sub-set has an upper (lower) bound, then given any element $x$ of the set there exists a maximal (minimal) element $x^*$ in the set such that $x$ is less (greater) than $x^*$.

[The terms that are underlined are defined in terms of the partial order relation].

**Lemma 2 (Extension of Measures):** Given a measure $\mu$ defined on a field $F$ of sets there exists one and only one extension $\mu^*$ of $\mu$ to the Borel-extension $F^*$ of $F$. 

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Corollary: If the two measures \( \mu \) and \( \nu \) agree on a field \( F \) of sets they necessarily agree on the Borel-extension \( F^* \) of \( F \).

Lemma 5: If the family \( \{ B_a \} \) of sub-fields of \( A \) be linearly ordered with respect to the inclusion relation then

\[
E = \bigcup B_a
\]

is a field of sub-sets.

We omit the proofs of the lemmas.

Now let \( B \) be a given sub-field of \( A \) and let \( E \) be the class of all sets \( E \in A \) such that \( E \) is independent of \( B \), i.e.

\[
P(EB) = P(E)P(B) \text{ for all } B \in B, E \in E \text{ and } P \in P.
\]

It is easy to check that \( E \) contains the null-set and the whole space and further that \( E \) is closed for complementation and countable disjoint unions. In case \( E \) is a \( \sigma \)-field there is nothing to prove in Theorem 1, as \( E \) is then the maximal sub-field for which we are searching. However, \( E \) is usually not a \( \sigma \)-field (see example 1).

Let \( \{ C_a \} \) be a family of sub-fields in \( E \) and be linearly ordered with respect to the inclusion relation and let

\[
C_0 = \bigcup C_a.
\]

Now, from Lemma 3, \( C_0 \) is a field of sub-sets of \( A \) and, since \( C_0 \subseteq E \), every member of \( C_0 \) is independent of \( B \). Choose and fix \( B \in B \) and \( P \in P \).

Consider the two measures \( P(AB) \) and \( P(A)P(B) \) defined for all sets \( A \in A \). These two measures agree over the field \( C_0 \) and hence, from the corollary to Lemma 2, they agree over the Borel-extension \( C_0^* \) of \( C_0 \).

Remembering that \( B \) and \( P \) were arbitrary members of \( B \) and \( P \) respectively, we now have
\[ P(AB) = P(A)P(B) \text{ for all } A \in \mathcal{C}^*, \ B \in \mathcal{B} \text{ and } P \in \mathcal{P}. \]

Thus, \( \mathcal{C}^* \) includes every \( \mathcal{C}_\alpha \) and is independent of \( \mathcal{B} \). The conditions of Lemma 1 are satisfied and hence the proof of Theorem 1 is complete.

We now turn our attention to Theorem 2. Let \( \mathcal{F} \) be the class of all sets \( F \in \mathcal{A} \) such that

\[ P_{\theta, \varphi}(F) \text{ is a function of } \theta \text{ only.} \]

Again, it is easy to check that \( \mathcal{F} \) contains the null-set and the whole space and is closed for complementation and countable disjoint unions. In example 2 we shall see that \( \mathcal{F} \) is usually not a sub-field of \( \mathcal{A} \).

Let \( \{ \mathcal{Q}_\alpha \} \) be a family of sub-fields in \( \mathcal{F} \) and be linearly ordered with respect to the inclusion relation and let

\[ \mathcal{Q}_0 = \bigcup \mathcal{Q}_\alpha. \]

As before \( \mathcal{Q}_0 \) is a field of sub-sets of \( X \). Let \( \mathcal{Q}_0^* \) be the Borel-extension of \( \mathcal{Q}_0 \).

We define the measure \( Q_\theta \) on \( \mathcal{Q}_0 \) as the restriction of \( P_{\theta, \varphi} \) on \( \mathcal{Q}_0 \). [Since \( \mathcal{Q}_0 \subset \mathcal{F} \), the measure \( Q_\theta \) on \( \mathcal{Q}_0 \) must be independent of \( \varphi \).]

From Lemma 2, for each \( \theta \in \Theta \), the extension \( Q_\theta^* \) of \( Q_\theta \) from \( \mathcal{Q}_0 \) to \( \mathcal{Q}_0^* \) is unique. From the corollary to Lemma 2, the two measures \( Q_\theta^* \) and \( P_{\theta, \varphi} \) must agree on \( \mathcal{Q}_0^* \).

In other words, the restriction of \( P_{\theta, \varphi} \) to the sub-field \( \mathcal{Q}_0^* \) is independent of \( \varphi \). Also, \( \mathcal{Q}_0^* \) includes every \( \mathcal{Q}_\alpha \). The conditions of Lemma 1 are satisfied and hence the proof of Theorem 2 is complete.
3. Examples and Comments.

The following two examples demonstrate that the maximal element is usually not unique.

**Example 1:** Let $X$ consist of the four points $a, b, c$ and $d$ and let $P$ consist of just one probability distribution namely the uniform distribution over the four points. Consider the three sub-fields $B, C_1$ and $C_2$ each consisting of four sub-sets of $X$:

- $B$ consists of $X$, $(a, b)$ and their complements,
- $C_1$ ....... $X$, $(a, c)$ ..............,
- $C_2$ ....... $X$, $(a, d)$ ..............

Here, each of $C_1$ and $C_2$ is independent of $B$ and each of them is a maximal such sub-field. Incidentally, $C_1$ and $C_2$ are also independent of each other.

**Example 2:** Let $X$ consist of the five points $a, b, c, d$, and $e$ and let the probability distribution over the five points be

Points: $a$ $b$ $c$ $d$ $e$

Probs: $1-\theta$ $\varphi$ $\varphi$ $\varphi(\frac{1}{2}-\varphi)$ $\varphi(\frac{1}{2}-\varphi)$

where $0 < \theta < 1$ and $0 < \varphi < \frac{1}{2}$.

The family $P$ of all sets whose probability does not involve $\varphi$ consists of 12 sets, that is

- $X$, $(a)$, $(b, d)$, $(b, e)$, $(c, d)$, $(c, e)$

and their complements.

Note that $P$ does not constitute a sub-field. There are two different maximal sub-fields in $P$, namely,

- $B_1$: consisting of $X$, $(a)$, $(b, d)$, $(c, e)$

and their complements.
and \( \mathbf{Q} \) consisting of \( x, \{a\}, \{b, e\}, \{c, d\} \)
and their complements.

Theorems 1 and 2 only establish the existence of maximal sub-fields in \( E \) and \( F \) respectively. It would be of some interest to develop general methods for proving the maximality of certain given sub-fields of \( E \) and \( F \). One such method, with very limited application, is given in Theorem 7 of [2].

Consider the problem where we have \( n \) independent observations \( x_1, x_2, \ldots, x_n \) on a real random variable \( x \) with cumulative distribution function of the form

\[
F\left( \frac{x - \varphi}{\theta} \right), \quad -\infty < \varphi < \infty, \quad 0 < \theta < \infty.
\]

where the function \( F \) is known and \( \theta \) and \( \varphi \) are the so-called scale and location parameters.

If \( y \) stand for the vector-valued statistic

\[
(x_1 - x_n, x_2 - x_n, \ldots, x_n - x_n)
\]

then the distribution of \( y \) does not involve the location parameter \( \varphi \). Is \( y \) a maximal such statistic? In the language of sub-fields, if \( C_y \) be the sub-field generated by \( y \) then is it true that \( C_y \) is maximal in the sense of Theorem 2? The author does not expect the answer to be 'yes' for all \( F \).

4. Some further problems

The sub-field \( B \subset A \) is said to be sufficient for the sub-field \( C \subset A \) if for every \( C \subset C \) there exists a \( B \)-measurable function \( f(x; C) \) mapping \( X \) into the real line such that

\[
P(BC) = \int_B f(x; C) \, dP(x) \text{ for all } P \in \mathcal{P}
\]

and \( B \subset B \).
In other words, \( B \) is sufficient for \( C \) if, for every \( C \in \mathcal{C} \), there exists a choice for the conditional probability (function) of \( C \) given \( B \) that serves for all \( P \in \mathcal{P} \).

Now, for a fixed \( B \), let us enquire about the family \( \{C\} \) of all subfields \( C \) such that \( B \) is sufficient for \( C \). Clearly, the minimum element of \( \{C\} \) is the trivial sub-field consisting of only the null-set and the whole space. Do there exist maximal elements in \( \{C\} \)?

Let \( G \) be the class of all sets \( G \in \mathcal{A} \) such that \( B \) is sufficient for \( G \) in the sense mentioned above, namely, for every \( G \in \mathcal{G} \) there exists a \( B \)-measurable \( f(x;G) \) such that

\[
P(GB) = \int f(x;G)dP(x) \text{ for all } P \in \mathcal{P}
\text{and } B \in B.
\]

The class \( G \) is similar to the classes \( E \) and \( F \) considered before in that \( G \) contains the null-set and the whole space and is closed for complementation and countable disjoint unions. As before, \( G \) is usually not a sub-field. The rest of the arguments in Theorems 1 and 2 will apply if we could prove a result of the following type:

"If \( B \) is sufficient for each member of a field \( \mathcal{C} \) of sets in \( \mathcal{A} \), then \( B \) is sufficient for the Borel extension \( \mathcal{C}^* \) of \( \mathcal{C} \)."

The above statement does not seem to be true in the generality stated above.

In the particular case where \( B \) is the trivial sub-field, the question posed above has a definite answer. For, in this case, \( B \) can be sufficient for \( G \) if and only if

\[
P(G) \text{ is the same for all } P \in \mathcal{P},
\]
and therefore, Theorem 2, or rather a particular case of it, namely, Theorem 1.

Fraser in [3] introduced the notation of partial sufficiency in the following manner:

If \( P = \{ P_{\theta, \varphi} \} \), \( \theta \in \Theta \), \( \varphi \in \Phi \), be a family of probability measures indexed by the two independent parameters \( \theta \) and \( \varphi \), then a sub-field \( \mathcal{B} \subseteq \mathcal{A} \) will be called \( \theta \)-sufficient for \( \mathcal{A} \) (or simply \( \theta \)-sufficient) if

1) the restriction of \( P_{\theta, \varphi} \) to \( \mathcal{B} \) does not depend on \( \varphi \) [i.e., \( \mathcal{B} \) is a sub-field of the type considered in Theorem 2.],

and 2) given any \( A \in \mathcal{A} \), there exists a choice (of the conditional probability function) of \( A \) given \( \mathcal{B} \) that does not depend on \( \theta \), i.e., for each \( \theta \in \Theta \) there exists a \( \mathcal{B} \)-measurable function \( f_\theta(x;A) \) such that

\[
P_{\theta, \varphi}(AB) = \int_B f_{\theta}(x;A) \, dP_{\theta, \varphi} \quad \text{for all } B \subseteq \mathcal{B}
\]

and all \( (\theta, \varphi) \).

Under what conditions does a \( \theta \)-sufficient sub-field exist? Does there exist an essentially minimum such sub-field?

As a final problem on the existence of minimal sub-fields consider the following:

Given two sub-fields \( \mathcal{B} \) and \( \mathcal{C} \), let \( \mathcal{B} \lor \mathcal{C} \) stand for the smallest sub-field that contains both \( \mathcal{B} \) and \( \mathcal{C} \).

Now, for a fixed \( \mathcal{B} \subseteq \mathcal{A} \), let us consider the family \( [\mathcal{C}] \) of all sub-fields \( \mathcal{C} \subseteq \mathcal{A} \) such that

\[
\mathcal{B} \lor \mathcal{C} = \mathcal{A}.
\]

Every \( \mathcal{C} \in [\mathcal{C}] \) may be called a complement of \( \mathcal{B} \). The family \( [\mathcal{C}] \) has \( \mathcal{A} \) as its maximum element. Does \( [\mathcal{C}] \) have minimal elements? The author expects the answer to be yes. It is easy to construct examples where there are several minimal complements to \( \mathcal{B} \).
References


