THE PETTIS-STIELTJES (STOCHASTIC) INTEGRAL

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1. INTRODUCTION

Various definitions for integrals of functions with values in an arbitrary Banach space $X$ (vector-valued integrals) have been given in at least three topologies: The weak and strong topologies on $X$ and pointwise in the scalar field topology. See, for example, [1], [4] and [5]. In each case conditions must be determined so that the integral exists as a well-defined vector in $X$.

The purpose of this paper is to define and exhibit some of the properties of a Stieltjes integral for vector-valued measures (functions) in the weak topology on $X$. As the definition is motivated by B. J. Pettis' definition of a Lebesgue integral in $X$, the integral will be called the Pettis-Stieltjes integral.

In Section 2, some properties of vector measures used in the analysis are mentioned. The basic definition of the Pettis-Stieltjes integral and a listing of some of the commonly indicated integral properties comprise Section 3. The main result of Section 4 is a representation for the Pettis-Stieltjes stochastic integral in the form of a generalized integration by parts formula. The latter entails the use of a duality formula and an unsymmetric Fubini theorem. In Section 5, a comparison with other stochastic integrals is noted as well as another condition for existence via the definition of a modified Stieltjes integral in the strong topology on $X$.

Examples are given in Section 6 and there is an appendix cataloging properties of the scalar valued modified Stieltjes integral. For the interval $T$ in the real line, $\mathbb{R}$, take $-\infty < a = \inf t < \sup t = b < \infty$.

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2. VECTOR MEASURES

This section contains some preliminary data on vector-valued measures, definitions of their variations and comparisons.

Consider the measure space \((T, \mathcal{Q})\), where \(T\) is an interval in \(\mathbb{R}\), the reals, and \(\mathcal{Q}\) is a sigma field of subsets of \(T\). Let \(\mu\) be a set function (measure) on \(\mathcal{Q}\) to the Banach space \(X\) over \(\mathbb{R}\).

**DEFINITION 2.1.** \(\mu : \mathcal{Q} \to X\) is strongly (weakly) countably additive if for \(\{A_n\}\) disjoint in \(\mathcal{Q}\)

\[
\mu(\bigcup_{n} A_n) = \sum_{n} \mu A_n
\]

where convergence is in the norm (weak) topology.

Three definitions for a function \(x : T \to X\) to be of bounded variation are given in Hille-Phillips, [4, p.60], and two are shown to be equivalent. Below, variation functions for the measure \(\mu\) are introduced and equivalence is extended to yet another definition from Dunford-Schwartz, [1, p.320]. The supremum is taken over all finite partitions of \(A \in \mathcal{Q}\) unless otherwise noted.

\(X^*, X^{**}\) denote the first and second dual spaces of \(X\).

**DEFINITION 2.2.**

i) The weak variation of \(\mu\) on \(A \in \mathcal{Q}\) with respect to \(x^* \in X^*\) is

\[
W_A(\mu, x^*) \equiv \sup_{n} \sum_{k=1}^{n} |x^*(\mu A_k)|.
\]
ii) The (total) weak variation of $\mu$ on $A \in \mathcal{Q}$ is

$$W_A(\mu) \equiv \sup\{ W_A(\mu, x^*) : \|x^*\| \leq 1 \}.$$ 

iii) The semi-variation of $\mu$ on $A \in \mathcal{Q}$ is

$$\|\mu\| (A) \equiv \sup \| \sum_{k=1}^{n} \alpha_k \mu A_k \|$$

where the supremum is also taken over all $|\alpha_k| \leq 1$.

iv) The variation of $\mu$ on $A \in \mathcal{Q}$ is

$$V_A(\mu) \equiv \sup \| \sum_{k=1}^{n} \mu A_k \|.$$ 

v) The strong variation of $\mu$ on $A \in \mathcal{Q}$ is

$$S_A(\mu) \equiv \sup \sum_{k=1}^{n} \| \mu A_k \|.$$ 

**Proposition 2.3.**

i) $V_A(\mu) \leq \|\mu\| (A) \leq W_A(\mu) \leq S_A(\mu)$

ii) $\|\mu\| (A) = \beta_1 V_A(\mu)$

$$W_A(\mu) = \beta_2 V_A(\mu), \quad \beta_1 \leq \beta_2, \quad \beta_j \in [1, 2].$$

See, also, Section 6.

**Proof:** Let $W_A(\mu, \alpha) \equiv \sup \sum_{k=1}^{n} |\alpha_k x^*(\mu A_k)|$ over all $|\alpha_k| \leq 1$ and $\|x^*\| \leq 1$, then $W_A(\mu, \alpha) = W_A(\mu)$

and adapting the argument in [4], get

$$\|\mu\| (A) \leq W_A(\mu, \alpha)$$
as well as $W_A(\mu) < 2V_A(\mu)$. The remainder of the inequalities are evident.

**Corollary 2.4.** The following are equivalent.

i) $\mu$ has finite (total) weak variation (is of weak bounded variation) on $A$.

ii) $\mu$ has finite semi-variation on $A$.

iii) $\mu$ has finite variation on $A$.

Moreover, the finiteness of any of the variation functions $V_T(\mu)$, $\|\mu\|(T)$ or $W_T(\mu)$ is equivalent to $\mu$ having either type of countable additivity when $\mathcal{Q}$ is the Borel field, $\mathcal{B}(T)$.

**Proposition 2.5.** The statements below are equivalent for $\mu$: $\mathcal{B}(T) \to X$:

i) $\mu$ is strongly countably additive on $\mathcal{B}(T)$.

ii) $\mu$ is weakly countably additive on $\mathcal{B}(T)$.

iii) $\mu$ is of weak bounded variation on $T$, for $\mu$ finite.

Statements (i) and (ii) are equivalent for arbitrary $\mathcal{Q}$.

**Proof:** (i) $\iff$ (ii) in [4].

Let $m(\cdot) = x[\mu(\cdot)]$, $x \in X^*$.

(ii) $\implies$ (iii): the scalar case result is known (Hahn decomposition) for $m(\cdot)$.

(iii) $\implies$ (ii): this follows from the correspondence between Lebesgue-Stieltjes measures and non-decreasing, bounded functions on $T$. Let $f(\cdot) = m(\cdot, \cdot]$, then $f$ is the difference of two such functions.

Note. There is no confusion about $\mu$ being of finite weak variation or finite (total) weak variation since they are, also, equivalent:

(iii) $\implies$ (i) $\implies$ $\|\mu\|(T) < \infty$, [DS, p.320], $\implies W_T(\mu) < \infty$. 
Let BV(T) be the space of functions of bounded variation on T under the supremum norm

\[ \|f(\cdot)\|_u = \sup_{t \in T} |f(t)|. \]

In Section 4, we will be interested in the case where the vector measure \( \mu \) is induced by a function \( x: T \rightarrow X \). \( x \) defines \( \mu \) on the field of half-open intervals by

\[ \mu((a, t]] = x(t) - x(a) \]

and when \( x \) is of weak bounded variation on \( T \), i.e., the scalar function \( x^* \) defines \( \mu \) on the field of half-open intervals by

\[ x^*[x(\cdot)] = g^*(\cdot) \in BV(T) \]

for all \( x^* \in X^* \), then \( \Delta g^*(\cdot) = x^*[\mu] \) is countably additive on the field and extends uniquely to \( dg^*(\cdot) \) countably additive on \( \mathcal{B}(T) \) by the correspondence mentioned in the above proof. So the vector measure \( \mu = dx \) is defined by the values \( x^*(dx) = dg^* \).

When dealing with the measure induced by a function \( x: T \rightarrow X \), we will always use \( Q = \mathcal{B}(T) \) and take the usual partitions \( \{t_{kn}\} \), \( k = 0, \ldots, n; n = 1, 2, \ldots \). All of the above remarks are valid; in particular, for \( V_A(x) \), use partitions \( \{(s_{kn}, t_{kn})\} \), for all finite collections of non-overlapping intervals. For example,

\[ W_A(x, x^*) = \sup \sum_{k=1}^n |\Delta x^*[x(t_{kn})]|. \]

When \( A \) is an interval with endpoints \( c \) and \( d \), write \( W_{cd} \) or \( V_{cd} \). If \( x: T \rightarrow X = \mathbb{R} \), all of the above reduce to the ordinary variation of the function \( x \).
3. THE PETTIS-STIELTJES INTEGRAL

The basic definition is motivated by the definition of the Pettis integral, see Hille-Phillips [4, p.77], and is made possible by the following proposition, which is similar to the ordinary case. Let \( f \) be measurable with respect to \((T, \mathcal{Q})\) and we say that \( f \) is weakly integrable with respect to \( \mu \) if \( f(\cdot) \in L_1(T, \mathcal{Q}, x^*[\mu(\cdot)]) \) for all \( x^* \in \mathcal{X} \), where \( \mu \) is (weakly) countably additive on \( \mathcal{Q} \).

**Proposition 3.1.** Let \( \mu \) be countably additive on \( \mathcal{Q} \) and \( f \) weakly integrable with respect to \( \mu \). Then there exists \( x^{**} \in \mathcal{X}^{**} \) such that

\[
x^{**}(x^*) = \int_T f(t) x^*[\mu(dt)]
\]

for all \( x^* \in \mathcal{X}^* \).

The scalar-valued integral on the right side is the ordinary Lebesgue integral.

**Proof.** Let \( I(x^*) = \int_T f(t) x^*[\mu(dt)] \) for \( x^* \in \mathcal{X}^* \). \( I \) is obviously linear and

\[
W_{\alpha}(\mu, x^*) \leq \beta \|x^*\| V_{\alpha}(\mu)
\]

for some \( \beta \in [1, 2] \) by Proposition 2.3. So

\[
|I(x^*)| \leq \int_T |f(t)| W_{\alpha}(\mu, x^*) \leq \|x^*\| \beta \int_T |f(t)| V_{\alpha}(\mu) .
\]

Therefore \( I \) is bounded and, hence, in \( \mathcal{X}^{**} \).
With the above as a justification, \( x^{**} \) may be set equal to the symbol \[
\int_T f(t)\mu(dt) \in X^{**}.
\]

More precisely, for each \( f \), there is an \( x^{**}_f \in X^{**} \) which may or may not be in the image of \( X \) under the natural mapping
\[
i: X \rightarrow i(X) \subset X^{**}.
\]

If this correspondence obtains, we may define the Pettis-Stieltjes integral.

**DEFINITION 3.2.** The scalar-valued function \( f \) on \((T, Q)\) is Pettis-Stieltjes (PS-) integrable with respect to \( \mu: Q \rightarrow X \), a Banach space, if for all \( A \in Q \), there exists \( x_A \in X \) such that
\[
x^*(x_A) = \int_A f(t) x^*[\mu(dt)]
\]
for all \( x^* \in X^* \). By definition
\[
x_A = \text{PS-}\int_A f(t) \mu(dt) \in X
\]
and write \( f \in \text{PS}(\mu) \).

For notational purposes, we will often write
\[
<f, \mu>_A \equiv \text{PS-}\int_A f(t) \mu(dt).\]

Let \( X \) be reflexive. Then if \( \mu \) is weakly countably additive, \( f \in \text{PS}(\mu) \) if and only if \( f \) is weakly integrable with respect to \( \mu \); hence, always for \( X = L_p \) over an arbitrary measure space, \( 1 < p < \infty \).
COROLLARIES 3.3.

i) The PS-integral is uniquely defined.

ii) $\mu$ must be weakly countably additive for the definition.

iii) $f_1, f_2 \in PS(\mu) \Rightarrow \alpha_1 f_1 + \alpha_2 f_2 \in PS(\mu), \quad \alpha_j \in \mathbb{R},$

$f \in PS(\mu_1), PS(\mu_2) \Rightarrow f \in PS(\beta_1 \mu_1 + \beta_2 \mu_2), \quad \beta_j \in \mathbb{R},$

and $<f, \mu> \text{ is a bilinear function for fixed } A \in \mathcal{Q}.$

iv) If $X = \mathbb{R},$ then $<f, \mu>$ reduces to the ordinary Lebesgue integral.

Proof. Use the definition and the fact that $x^*(x) = x^*(y)$ for all $x^* \in X^*$
implies $x = y.$

PROPOSITION 3.4. Let $f \in PS(\mu)$ and $\psi(A) = <f, \mu>,$ then $\psi: \mathcal{Q} \rightarrow X$
is (strongly) countably additive.

Proof. Let $\{A_n\}$ disjoint in $\mathcal{Q}.$ Then for $x^* \in X^*$

$$x^*[(\cup A_n)] = <f, x^*(\mu)>_{\cup A_n} = \sum <f, x^*(\mu)>_{A_n} = \sum x^*[(\cap A_n)].$$

So $\psi$ is countably additive by Proposition 2.5.

REMARK. Call $\phi: \mathcal{Q} \rightarrow X$ continuous with respect to $\psi: \mathcal{Q} \rightarrow X,$ if for
all $\epsilon > 0,$ there exists $\delta > 0$ such that $||\phi(A)|| < \epsilon$ whenever $||\psi(A)|| < \delta.$

Then $\psi$ is continuous with respect to $\mu$; in fact, $\psi$ is continuous with
respect to $x^*(\mu)$ for any $x^* \in X^*$:

Let $x^* \in X^*.$ If $x^*[(\cap A)] = 0$ then $\psi A = 0$ and the result follows by
[4, p.76].
PROPOSITION 3.5. Let $L$ be a bounded, linear operator on $X$. If $f \in \text{PS}(\mu)$, then $f \in \text{PS}(L\mu)$ and for $A \in \mathcal{Q}$

$$L <f, \mu>_A = <f, L\mu>_A.$$  

Proof. Let $L^*$ be the adjoint of $L$. For $x^* \in X^*$, there is a unique $y^* \in X^*$ given by $y^* = L^*x^*$ and $y^*(\mu) = x^*(L\mu)$ is countably additive. Thus there exists

$$<f, x^*(L\mu)> = <f, y^*(\mu)> = y^*(<f, \mu>) = x^*(L<f, \mu>).$$

Hence, $<f, L\mu> = L<f, \mu>$.

REMARK. For $f \in \text{PS}(\mu)$, $\nu = <f, \mu>$ has finite variation on $\mathcal{Q}$ but does not necessarily have finite strong variation. See Section 6.

PROPOSITION 3.6. Let $f \in \text{PS}(\mu_n)$, $n = 1, 2, \ldots$.

i) If $\mu_n \rightarrow \mu \rightarrow 0$ in weak variation then $f \in \text{PS}(\mu)$ and for $A \in \mathcal{Q}$

$$<f, \mu>_A = \lim_{n \rightarrow \infty} <f, \mu_n>_A \quad \text{(weak)}.$$  

ii) $\mu_n \rightarrow \mu$ in weak variation in (i) is not sufficient.

Proof. i) $\mu$ is countably additive since

$$W_A(\mu, x^*) \leq W_A(\mu_n - \mu, x^*) + W_A(\mu_n, x^*)$$

and $f \in \text{PS}(\mu)$.

$$| <f, x^*(\mu_m - \mu_n)>_A | \leq | f | \cdot W_{\text{at}}(\mu_m - \mu_n, x^*)_A \rightarrow 0$$

as $m, n \rightarrow \infty$, so there exists a weak limit of $<f, \mu_n>$ and
\[ x^*\langle f, \mu_n \rangle = \langle f, x^*(\mu_n) \rangle \rightarrow \langle f, x^*(\mu) \rangle. \]

The left side converges to \( x^*(\lim \langle f, \mu_n \rangle) \) and the right side is \( x^*(\langle f, \mu \rangle). \)

ii) Let \( \mu_n \equiv -\mu \). \( W(\mu_n, x^*) \rightarrow W(\mu, x^*) \) but \( W(\mu_n - \mu, x^*) = 2W(\mu, x^*) \not\to 0 \) and \( \langle f, x^*(\mu_n - \mu) \rangle \not\to 0. \)

**PROPOSITION 3.7.** Let \( f_n \in PS(\mu), n = 1, 2, \ldots \), and either of the following hold:

i) \( f_n \rightarrow f \) in \( \| \cdot \|_\mu \).

ii) \( f_n \rightarrow f \) pointwise, \( |f_n| \leq g \) weakly integrable with respect to \( \mu \).

Then for \( A \in Q \)

\[ \langle f, \mu \rangle_A = \lim_{n} \langle f_n, \mu \rangle_A \quad \text{(weak)}. \]

**Proof.** \( f_n, f \) are bounded by an integrable function; the argument in 3.6. applies and the inequality

\[ |\langle f_n - f, x^*(\mu) \rangle_A| \leq \| f_n - f \|_\mu \cdot W_A(\mu, x^*) \]

is evident.

Note that it is sufficient for the convergences and inequality in (i) and (ii) to be (weak) \( \mu - a.e. \), of course.

An acceptable strong definition implies the weak definition in much the same way that existence of the Bochner (ordinary) integral implies existence and equality of the Pettis integral. For example, using a definition in Dunford-Schwartz [1, p.323],

**DEFINITION 3.8.** A scalar-valued measurable function \( f \) is said to be integrable if there exists a sequence of simple functions \( \{f_n\} \) such that
i) \( f_n \to f \) \( \mu \) - a.e. and

ii) \( \{ \int_A f_n(t) \, \mu(dt) \} \) converges in norm for \( A \in \mathcal{Q} \), where

\[
\int_A f_n(t) \, \mu(dt) = \Sigma \alpha_k \mu A_k \quad \text{and} \quad \{A_k\} \text{ partitions } A.
\]

The limit in (ii) is defined to be \( \text{DS-} \int_A f(t) \, \mu(dt) \).

**REMARK.** The existence of the strong integral \( \text{DS-} \int_A f(t) \, \mu(dt) \) implies the existence of the (weak) Pettis-Stieltjes integral \( \text{PS-} \langle f, \mu \rangle_A \) and the integrals coincide:

\[
x^*(\int_A f_n \, d\mu) = \sum \alpha_k x^*(\mu A_k) = \int_A f_n x^*(d\mu) = \int_A f x^*(d\mu),
\]

the left side converges to \( x^*(\text{DS-} \int_A f \, d\mu) \) and the right side is \( \text{PS-} \langle f, x^*(\mu) \rangle_A \).

**PROPOSITION 3.9.** Let \( f, g \) be scalar-valued functions on \( (T, \mathcal{Q}) \) and \( \nu(A) = \langle g, \mu \rangle_A \). Then \( f \in \text{PS}(\nu) \) if and only if \( fg \in \text{PS}(\mu) \) and for \( A \in \mathcal{Q} \)

\[
\langle f, \nu \rangle_A = \langle fg, \mu \rangle_A.
\]

**Proof.** Let \( f \in \text{PS}(\nu) \) and \( x^* \in X^* \). Then there exists \( \langle f, x^*(\nu) \rangle_A \) and \( \{f_n\} \) simple such that \( f_n \to f \), \( x^*(\mu) \) - a.e. and in \( L_1(T) \). (If necessary, consider the non-negative parts of \( f, g \) and \( x^*(\mu) \) and use the linearity.)

\[
f_n g \to fg \text{ a.e. and}
\]

\[
\langle f_m g - f_n g, \mu \rangle_A = \langle f_m - f_n, \nu \rangle_A.
\]

Hence \( \{f_n g\} \) is Cauchy in \( L_1(T) \) with respect to \( x^*(\mu) \) and there exists \( h^* \in L_1(T) \) such that \( f_n g \to h^* \). Consequently, \( h^* = fg \) a.e. and...
\[ \langle f, x^*(\nu) \rangle_A = \langle fg, x^*(\mu) \rangle_A. \]

If \( fg \in \text{PS}(\mu) \), there exists \( \{f_n\} \) simple such that \( f_n \leq |f| \) and \( f_n \to f \) a.e. So

\[ |\sup f_n, x^*(\nu)\rangle_A | \leq |f|, W_{at}(\mu, x^*) \]

and \( f \in \text{PS}(\nu) \).
In this section, consider the vector measure \( \mu \) induced by a random function \( \mathcal{X} : T \rightarrow \mathcal{X} \), where \( \mathcal{X} = \mathcal{X}(\Omega, \mathcal{F}, \mathbb{P}) \) is a Banach space of random variables over a probability space. If \( \mathcal{X} = L_1(\Omega) \), then there exists \( E|x(t)| < \infty \) for all \( t \in T \). Recall the representation for the dual space of \( L_1(\Omega) \): given \( x(t) \in L_1(\Omega) \) and \( x^* \in L_1^*(\Omega) \), there exists \( \xi^* \in L_\infty(\Omega) \) such that

\[
x^*[x(t)] = E[x(t)\xi^*].
\]

The general definition for the PS-integral becomes in this case,

**DEFINITION 4.1.** The scalar-valued function \( f \) on \( (T, \mathcal{Q}) \) is PS-integrable with respect to the random function \( x = \{x(t): t \in T\} \) with values in \( L_1(\Omega) \), if for all \( A \in \mathcal{Q} \), there exists \( x_A \in L_1(\Omega) \) such that

\[
E[x_A\xi^*] = \int_A f(t) \, dE[x(t)\xi^*]
\]

for all \( \xi^* \in L_\infty(\Omega) \). By definition

\[
x_A = \text{PS-} \int_A f(t) \, dx(t) \in L_1(\Omega).
\]

To have the induced measure \( \mu \) be countably additive, it suffices to restrict the scalar function \( g^*(\cdot) \equiv E[x(\cdot)\xi^*] \) to be in \( BV(T) \), for all \( \xi^* \in L_\infty(\Omega) \). Make this assumption in the sequel.

The above is a definition of a stochastic integral in the weak topology of \( \mathcal{X} = L_1(\Omega) \). We will also need another stochastic integral defined point-wise in the scalar topology.
Let $x$ and $y$ be random functions on $(T, \mathcal{Q})$ to $X(\Omega, F, \mathbb{P})$.

**DEFINITION 4.2.** The sample path stochastic integral of $y$ with respect to $x$ over $A \in \mathcal{Q}$ exists and is denoted by

$$\text{SP-} \int_A y(t, \cdot) \, dx(t, \cdot),$$

if the scalar-valued integrals

$$(\text{type}) - \int_A y(t, \omega) \, dx(t, \omega)$$

exists for almost all $\omega \in \Omega$, where the type may be one of the Riemann-Stieltjes (RS-) integral definitions or the Lebesgue-Stieltjes (LS-) integral.

Conditions for existence and properties of the SP-integral are discussed in [5]. In both of the above definitions, the integrals are defined on $\Omega$ a.s. with respect to $\mathbb{P}$.

Assume that $x : (T, \mathcal{B}(T)) \rightarrow L_1(\Omega, F, \mathbb{P})$ is a non-trivial product measurable random function and $T$ is a finite interval.

**THEOREM 4.4.** For $f \in \text{BV}(T)$ and $x$ of weak bounded variation on $T$, the Pettis-Stieltjes integral exists and has the representation

$$\text{PS-} \int_A f(t) \, dx(t) = \text{SP-} \int_A x(t, \cdot) \, dm_f(t) \text{ a.s.}$$

where $m_f(\cdot) \in \text{CA}(T)$ the countably additive scalar-valued set functions on $\mathcal{B}(T)$. 
The integral on the right side is the sample path Lebesgue-Stieltjes type.

By the remark following Definition 3.8, the formula is true for strong integrals.

**Corollary 4.5.** When it exists,

\[
\int_T f(t) \, dx(t) = \int_T x(t, \cdot) \, dm_f(t).
\]

The proof of the theorem requires a few preliminary results.

Let \( BD_1(T) \) be the space of bounded, scalar-valued functions on \( T \) with at most discontinuities of the first kind. For \( f \in BD_1(T) \) and \( g \in BV(T) \), the modified Stieltjes integral \( MS-\int_T f(t) \, dg(t) \) may be defined; it is a generalization of the Riemann-Stieltjes integral and is discussed in the appendix.

**Definition 4.6.** Let \( \mathcal{C}(T) \) be the field of finite unions of (finite) open intervals and points in \( T \). For \( f \in BD_1(T) \), define \( m_f(\cdot) \) on \( \mathcal{C}(T) \) by

\[
m_f(C) = MS-\int_T f(t) \, dI_C(t).
\]

Let \( BA(T) \) be the space of bounded, finitely additive, scalar-valued set functions on \( T \).

**Lemma 4.7.**

(i) When \( f \in BD_1(T) \), \( m_f \in BA(T) \) on \( \mathcal{C}(T) \) and

\[
m_f((c, d)) = f(c^+) - f(d^-), \quad m_f(\{c\}) = f(c^-) - f(c^+).
\]
(ii) When \( f \in BV(T) \), \( m_f \in CA(T) \) on \( \mathcal{B}(T) \) and is bounded.

**Proof:** Let \( C \in \mathcal{C}(T) \). \( C = \bigcup_{j=1}^{m} (c_j, d_j) \cup \bigcup_{k=1}^{n} \{e_k\} \), disjoint. (i) follows from Proposition A.6 (appendix), Definition 4.9 and the form of \( C \).

\[
\sum_{j=1}^{m} |f(c_j^+) - f(d_j^-)| + \sum_{k=1}^{n} |f(e_k^-) - f(e_k^+)| \leq V_T(f) < \infty ;
\]

hence, \( \sup \{ |m_f(C)| : C \in \mathcal{C}(T) \} \) is finite. Also \( f = f_1 - f_2, f_k \) increasing, \( k = 1, 2 \). Therefore,

\[
m_f = m_{f_1} - m_{f_2}
\]

where \( m_{f_k} \in CA(T) \). As a result, \( m_f \) may be (uniquely) extended to the sigma field generated by \( \mathcal{C}(T) \), by the Carathéodory Extension theorem, but this is \( \mathcal{B}(T) \).

**PROPOSITION 4.8.** For \( f \in BV(T) \) and \( x \) of weak bounded variation on \( T \), a duality formula is obtained:

\[
\int_T f(t) \, dg^*(t) = \int_T g^*(t) \, dm_f(t)
\]

for all \( \xi^* \in L_\infty(\Omega) \), where \( m_f \in CA(T) \) and \( g^*(\cdot) = E[x(\cdot) \xi^*] \).

**Proof:** We refer to Proposition A.8 in the appendix for an integration by parts result for modified Stieltjes integrals.

\[
\text{MS-} \int_T f \, dg = \text{MS-} \int_T gdf + [fg]_a^b + \sum_{x} [f(x^-)(g(x)-g(x^-)) - f(x)(g(x^+)-g(x^-)) + f(x^+)(g(x^+)-g(x))] + [fg]_a^b
\]

where the sum is over the (common) discontinuities of \( f \) and \( g \).
Letting $g$ be $g^*$, our desired result obtains if we can show that
\[ \int_T g(t) \ dm_f(t) \] expands to become the right hand side of the above equation.

From the definition,
\[ \int_T g(t) \ dm_f(t) = \lim_{D \to \infty} \sum_{k=1}^{n} g(t_k^i) \Delta m_f(t_k) \]
with partitions from $\mathcal{P}(T)$ and the $t_k^i$ are arbitrary interior points.

For $t_k^i \in (t_{k-1}, t_k)$: $\Delta m_f(t_k) = m_f((t_{k-1}, t_k)) = f(t_{k-1}+) - f(t_{k-1})$.

For $t_k^i = t_k$: $\Delta m_f(t_k) = m_f([t_k]) = f(t_k^-) - f(t_k^-)$.

Hence $\sum_{D} g(t_k') \Delta m_f(t_k') =$
\[ \sum_{\text{intervals}} \sum_{D} g(t_k') [f(t_{k-1}^i) - f(t_{k-1})] + \sum_{D} g(t_k') [f(t_{k-1}) - f(t_{k-1})] \]
and writing the first square bracket as
\[ - \left( f(t_{k-1}) - f(t_{k-1}) \right) + \left( f(t_{k-1}) - f(t_{k-1}) \right) - \left( f(t_{k-1}) - f(t_{k-1}) \right) \]
we get the following sums
\[
S_1 = \sum_{D} g(t_k') \Delta (-f)(t_k'), \\
S_2 = \sum_{D} g(t_k') \left[ f(t_{k-1}) - f(t_{k-1}) \right] + \left[ f(t_{k-1}) - f(t_{k-1}) \right], \\
S_3 = \sum_{D} g(t_k') \left[ f(t_{k-1}) - f(t_{k-1}) \right].
\]

$S_2, S_3$ are non-zero only for the discontinuities of $f, \{d_j\}$, a countable set. Therefore, including these in a sequence of increasing partitions and taking the limit,
\[
S_1 \to - \int_T g(t) df(t), \quad (+[fg]_a^b \text{ for } f \text{ continuous at endpoints}) \\
S_2 \to \lim_{j=1}^{\infty} \left[ g(d_j^-) \{ f(d_j) - f(d_j^-) \} + g(d_j^+) \{ f(d_j^+) - f(d_j) \} \right], \\
S_3 \to \lim_{j=1}^{\infty} g(d_j) \left[ f(d_j) - f(d_j^+) \right].
\]
If the endpoints appear in $S_j$, we get

$$
\sum_{k} g(t_k)[f(t_k^-) - f(t_k^+)] = g(a)[-f(a)] + g(b)[f(b)] = [fg]_a^b.
$$

Adding, we get

$$
\int_T g(t) \, df(t) + [fg]_a^b
$$

$$
+ \sum_{j=1}^n [f(d^-_j)(g(d^-_j) - g(d^-_j)) - f(d^+_j)(g(d^+_j) - g(d^-_j)) + f(d^-_j)(g(d^-_j) - g(d^-_j))]
$$

**LEMMA 4.9.** For $f \in BV(T)$, the Lebesgue integral

$$
\int_T x(t, \cdot) \, dm_f(t)
$$

exists.

**Proof.** Use the Fubini theorem and the fact that $Ex(\cdot)$ is bounded, uniformly in $t$. ( $x$ is product measurable with respect to

$(T \times \Omega, \mathcal{B}(T) \otimes F, \lambda \otimes P)$, where $\lambda$ is Lebesgue measure.)

**PROPOSITION 4.10.** Let $f \in PS(x)$, then

(i) An unsymmetric Fubini theorem obtains,

$$
E\left[ \int_T f(t) \, dx(t) \right] = \int_T f(t) \, dEx(t);
$$

in fact,

$$
E\left[ \int_T f(t) \, dx(t) \xi^* \right] = \int_T f(t) \, dE[x(t) \, \xi^*]
$$

for all $\xi^* \in L_\infty(\Omega)$.

(ii) Moreover, if $f \in BV(T)$, the above also coincide with

$$
E\left[ \int_T x(t, \cdot) \, dm_f(t) \xi^* \right] = \int_T E[x(t) \xi^*] \, dm_f(t)
$$

for all $\xi^* \in L_\infty(\Omega)$. 
Proof. The equality in (i) follows since \( f \in \text{PS}(x) \) and that in (ii) from Lemma 4.9 and the ordinary Fubini theorem. Using the duality formula in Proposition 4.8,

\[
\int_T f(t) \, dE[x(t) \xi^*] = \int_T E[x(t) \xi^*] \, dm_f(t) = E\left[ \int_T x(t, \cdot) \, dm_f(t) \xi^* \right]
\]

\(<\Rightarrow\) there exists

\[
\lim_{D \to \text{D}} E\left[ \sum_{j=1}^{m} f(t_j) \Delta x(t_j) \xi^* \right] = \lim_{D \to \text{D}} E\left[ \sum_{j=1}^{m} f(t_j) \Delta E[x(t_j) \xi^*] \right]
\]

\[= E\left[ \int_T x(t, \cdot) \, dm_f(t) \xi^* \right],
\]

where the limit exists as a MS-integral. Thus

\[\lim_{D \to \text{D}} E[y_D \xi^*] = E[y \xi^*]\]

for all \( \xi^* \in L_\infty(\Omega) \); i.e., \( \lim_{D \to \text{D}} x^*(y_D) = x^*(y) \) for all \( x^* \in X^* \). Hence the weak limit of \( y_D \) exists and is \( y \);

\[\text{weak lim}_{D \to \text{D}} E[f(t_j) \Delta x(t_j)] = \int_T x(t, \cdot) \, dm_f(t) \in L_1(\Omega).
\]

Denoting the left limit by \( x_T = \int_T f(t) \, dx(t) \), we get the desired relation

\[E[x_T \xi^*] = \int_T f \, dE[x \xi^*] = \int_T E[x \xi^*] \, dm_f = E\left[ \int_T x \, dm_f \xi^* \right].
\]

**Definition 4.11.** Define \( L_1(x) \) to be the closure, in the norm topology, of the (linear) span of \( \{x(t) : t \in T\} \).

**Proof of Theorem 4.4.**

\( x = (x(t) : t \in T) \neq \{0\} \Rightarrow \dim L_1(x) \geq 1 \Rightarrow L_1(x) \) is a non-trivial
Banach space in \( L_1(\Omega) \Rightarrow L_1^*(x) \) is total. Existence of the PS- \( \langle f, x \rangle \) is assured; see, for example, Corollary 5.5. By Proposition 4.10,

\[
x^*\left( \int f \, dx \right) = \int f \, dx^*(x) = \int x^*(x) \, dm_f = x^*\left( \int x \, dm_f \right)
\]

for all \( x^* \in X^* \). Consequently,

\[
\text{PS-} \int f \, dx = \text{SP-} \int x \, dm_f
\]

in \( L_1(x) \), which means a.s.

In effect, the Pettis-Stieltjes stochastic integral of \( f \) with respect to the random function \( x \) is represented by an integration by parts formula. The following discussion motivates Theorem 4.4, indicates the origin of Definition 4.6 and outlines the argument supporting the view of the formula as a representation.

By the definition of the PS- integral, existence requires

\[
x_f^{**} \in i[L_1(\Omega)] \subset L_1^{**}(\Omega).
\]

Identifying the respective dual spaces,

\[
L_1(\Omega) \to L_1^*(\Omega) \cong L_\infty(\Omega) \to L_\infty^*(\Omega) \cong BA(\Omega).
\]

where the equivalences are isometric isomorphisms. Our interest, however, is in the curve \( x = \{x(t): t \in T\} \) lying in the space \( L_1(\Omega) \) and we want to characterize the functionals

\[
E[\cdot \xi^*], \, \xi^* \in L_\infty(\Omega)
\]

which act on \( x(t), \, t \in T \). Rather than considering functionals on \( L_1(\Omega) \), we look at \( E[\cdot \xi^*] \) on the space generated by the curve \( x, \, L_1(x) \).
$L_1(x)$ is a closed, linear subspace of $L_1(\Omega)$ and, consequently, is a Banach space under the relative topology of the $L_1(\Omega)$ norm, $E|\cdot|$. The functionals on $L_1(x)$ look like $g^* + L_1^*(x)$, where $g^*(\cdot) = E[x(\cdot)\xi^*]$ and $L_1^*(x)$ is the annihilator of $L_1(x)$ in $L_1^*(\Omega)$, [1, p.72]. On $L_1(x)$, this is just $E[x(\cdot)\xi^*]$ which is in $BV(T)$.

Since the operator topology on $L_1^*(x)$ is the induced $\|\cdot\|_\infty$ topology on $L_\infty(\Omega)$, consider the $\|\cdot\|_u$ topology on $BV(T)$ and identify $(BV(T), \|\cdot\|_u)^*$. (Note: This is not necessarily an exact isometric isomorphism; the latter holds only in a special case.)

We recall the definition of $BD_1(T)$ and the characterization of its dual space by Hildebrandt, [2, p.873]. (The result allows $T$ to be $\mathbb{R}$.)

**Proposition 4.12.** $(BD_1(T), \|\cdot\|_u)^* \cong BA(T)$, where the set functions in $BA(T)$ are defined on the field, $\mathcal{C}(T)$, of subsets of $T$ and the correspondence is an isometric isomorphism.

$BV(T) \subset BD_1(T)$, so the characterization may be used for our purposes; in fact, there is an improvement due to the form of the functionals

$$x_f^{**}(\cdot) = \int_T f(t) \, d(\cdot)$$

on $(BV(T), \|\cdot\|_u)$.

Now we see exactly the motivation of Definition 4.6 since

$$m_f(C) = x_f^{**}(I_C) = \int_T f(t) \, dI_C(t) .$$

Combining this last result with the duality formula, the following framework is established for the correspondences between the various spaces used in the analysis:
i) $x(\cdot) \in \mathbb{E} = \mathbb{E}(T, L_1(\Omega))$, the space of random functions on $T$ to $L_1(\Omega)$ with $E(x(\cdot)) \in BV(T)$. $g^*(\cdot) = E[x(\cdot)] = \xi^*[x(\cdot)] \in BV(T)$, where $\xi^* \in L_\infty(\Omega)$.

\[
\begin{array}{ccc}
L_1(\Omega) & \xrightarrow{\xi^*} & R \\
\uparrow x & & \downarrow g^* \\
T
\end{array}
\]

ii) $\mathbb{E} \overset{\eta}{\rightarrow} BV(T)$:

$\eta[x(\cdot)] = g^*(\cdot)$ by $\eta[x(t)] = g^*(t) = \xi^*[x(t)]$.

iii) $\overset{\xi}{\mathbb{E}} \overset{g^{**}}{\rightarrow} R$:

$g^{**}(\cdot) = \int_T f d \xi \in BV^*(T) \overset{\xi^*}{=} CA(T)$. $(g^{**} \circ \eta)[x(\cdot)] = g^{**}[g^*(\cdot)] = \int_T g^*(t) \, dm_f(t) = \int_T \xi^*[x(t)] \, dm_f(t)$.

Thus $g^{**} \circ \eta : \mathbb{E} \rightarrow R$ and $g^{**} \circ \eta$ is a bounded, linear functional on $\mathbb{E}$, $g^{**} \circ \eta \in \mathbb{E}^*$.

iv) $\overset{\xi}{\mathbb{E}} \overset{\xi^*}{\rightarrow} L_\infty(\Omega)$:

$\zeta[x(\cdot)] = \xi^*$, where $\xi^*$ defines $g^*$.

v) $\overset{\xi^*}{\mathbb{E}} \overset{\xi^{**}}{\rightarrow} R$:

$\xi^{**}(\cdot) = \int_T f d \xi \in L_\infty^{**}(\Omega) \overset{\xi^*}{=} BA(\Omega)$. $(\xi^{**} \circ \zeta)[x(\cdot)] = \xi^{**}[\xi^*[x(\cdot)]] = \int_T f(t) \, d\xi^*[x(t)] = \int_T f(t) \, dg^*(t)$.

Hence $\xi^{**} \circ \zeta : \mathbb{E} \rightarrow R$ and $\xi^{**} \circ \zeta$ is a bounded, linear functional on $\mathbb{E}$, $\xi^{**} \circ \zeta \in \mathbb{E}^*$.

Comparing the above under the appropriate conditions, we get the representation $\xi^{**} \circ \zeta = g^{**} \circ \eta$ or, symbolically, the duality formula from 4.8, $f \circ \xi^* = m_f \circ \xi^*$. The relationship is illustrated in Figure 1.
We record the following from [4, p.77].

**DEFINITION 4.13.** The function $x$ on $(T, Q)$ to the Banach space $\mathcal{X}$ is Pettis-integrable if for all $A \in Q$, there exists $x_A \in \mathcal{X}$ such that

$$x^*(x_A) = \int_A x^*[x(t)] \, d\lambda(t)$$

for all $x^* \in \mathcal{X}^*$, where the integral on the right is the scalar-valued Lebesgue integral. By definition

$$x_A = P-\int_A x(t) \, d\lambda(t).$$

The result of Theorem 4.4 holds for the Pettis integral.

**COROLLARY 4.14.** Suppose $x$ is of weak bounded variation and $f \in PS(x)$. Then

$$PS-\int_A f(t) \, dx(t) = P-\int_A x(t) \, dm_f(t).$$

**Proof.** Let $x_A = PS-\langle f, x \rangle$. Then for $x^* \in \mathcal{X}^*$

$$x^*(x_A) = \int_A f dx^*(x) = \int_A x^*(x) \, dm_f = x^*(P-\int_A x \, dm_f)$$

by the duality formula. (Existence of the Pettis integral is insured by existence of the sample path integral, Lemma 4.9, and the usual Fubini theorem.)

**REMARK.** Let $T = [0, 1]$ and $f \in BD_1(T)$, but not in $BV(T)$, be defined by

$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-1} I_{A_n}(t), \quad t \neq 0; \quad f(0) = 0.$$ 

where $A_n = (\frac{1}{n+1}, \frac{1}{n})$. Then

$$V_T(f) \geq \sum_{k=1}^{n} \frac{1}{n} \left| f\left(\frac{1}{k}\right) - f\left(\frac{1}{k+1}\right) \right| = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{1}{k} + \frac{1}{k+1}\right) \to \infty.$$
Therefore, even for a random function as simple as \( x(t) = K + 0, \)
\[ \int_T x \text{d}m_f = \infty. \] In fact, should we restrict \( f \) to be continuous on \( T \) but
not in \( \text{BV}(T) \), we can't define \( \text{SP-} \int_T x(t, \cdot) \text{d}m_f(t) \) for, say, \( x \) being
the Brownian motion process, since the paths are a.s. not of bounded variation.
As a result, \( f \in \text{BV}(T) \) is a necessary hypothesis for the representation.

Although the representation for arbitrary continuous functions cannot
be obtained in general, a strong approximation is available.

**PROPOSITION 4.15.** Let \( f \) be continuous on \( T \). For \( \varepsilon > 0 \), there
exists a polynomial \( p \) on \( T \) such that
\[
\left| \int_T f(t) \text{d}x(t) - \int_T x(t, \cdot) \text{d}p(t) \right| < \varepsilon.
\]

**Proof.** The Weierstrass approximation theorem insures the existence of a
polynomial \( p \) with \( \|f - p\|_u < \varepsilon/4V_T(x) \). Since \( p \in \text{BV}(T) \), the integrals
exist and the representation holds for \( \text{PS-} \int_T p \text{d}x \). Let \( \|x^*\| \leq 1 \) and
\[
y = \int_T f \text{d}x - \int_T p \text{d}x,
\]
then using Proposition 2.3
\[
\left| x^*(y) \right| \leq \int_T \left| f - p \right| \text{d}W_{at}(x, x^*) < \frac{\varepsilon}{2},
\]
which means that \( \|y\| < \varepsilon \).
5. OTHER PROPERTIES

Restricting $x$ to be of weak bounded variation on $T$ is, in a sense, a weakest possible assumption, since we shall usually want to integrate, at the least, all continuous functions.

**Proposition 5.1.** If $\text{PS-} \int_T f(t) \, dx(t)$ exists for all $f$ continuous on $T$, then $x$ is of weak bounded variation on $T$.

**Proof.** $\text{MS-} \int_T f(t) \, dx^*[x(t)]$ exists for all $x^* \in X^*$ and equals $\text{RS-} \int_T f(t) \, dx^*[x(t)]$ for $f$ continuous, since the oscillation over $(r, t)$ and $[r, t]$ is the same. (See the appendix.) For RS-integrals, the proposition is known, [3, p.271]. So $x^*[x(\cdot)] \in \text{BV}(T)$, for all $x^* \in X^*$.

Let $\text{ID-} \int_T f(t) \, dx(t)$ be the well-known Itô-Doob stochastic integral in $L_2(\Omega, F, P)$.

**Proposition 5.2.** When the following stochastic integrals exist, the $\text{PS-} \int_T f(t) \, dx(t)$ exists and coincides with them:

i) $\text{SP-} \int_T f(t) \, dx(t, \cdot)$, $f$ bounded, LS-type in $L_1(\Omega)$.

ii) $\text{ID-} \int_T f(t) \, dx(t)$, where $\text{Ex}(\cdot) \in \text{BV}(T)$.

**Proof.** When $y = \text{SP} \int_T f \, dx$ exists, $f$ is measurable and $x$ is a.s. of bounded variation on $T$, hence $\text{Ex}(\cdot) \in \text{BV}(T)$. See [5]. Let $\xi^* \in L_\infty(\Omega)$,

$E[y \, \xi^*] = \int_T f(t) \, dE[x(t) \, \xi^*]$, hence $y = \text{PS-} \int_T f \, dx$. The argument for the ID-integral is similar to the remark following Definition 3.8.
Returning to the strong topology on $X$, the standard definition of a Riemann-Stieltjes integral in a Banach space, [4, p.62], may be slightly generalized.

**DEFINITION 5.3.** If \( \lim D \sum_{k=1}^{n} f(t'_{k}) \Delta x(t_{k}) \) exists in the norm topology with \( t'_{k} \) arbitrary in \((t_{k-1}, t_{k})\) and partitions are successively finer, then denote the limit by

\[
\text{MS-} \int_{T} f(t) \, dx(t)
\]

**PROPOSITION 5.4.** Let \( f \in \text{BD}_{1}(T) \) and \( x: T \to X \) such that \( x \) is of weak bounded variation on \( T \). Then \( \text{MS-} \int_{T} f(t) \, dx(t) \) exists (in the norm topology).

**Proof.** For \( \epsilon > 0 \), select a partition \( D_{\epsilon} \) of \( T \) such that \( \text{osc}(f) < \epsilon/4M \) over any open subinterval of any \( D \supset D_{\epsilon} \), where \( M = \nu_{T}(x) < \infty \). Recall that \( W(x, x^{*}) \leq 2M \| x^{*} \| \) for \( x^{*} \in X^{*} \) and let \( D, D' \supset D_{\epsilon} \).

\[
\| x^{*}[\sum_{D} f(t'_{j}) \Delta x(t_{j}) - \sum_{D'} f(t'_{k}) \Delta x(t_{k})] \| \leq \frac{\epsilon}{2} \sum_{D \cup D' \ j,k} \text{osc}(f) \| \Delta x^{*}[x(t_{j})] \| < \| x^{*} \| \epsilon / 2
\]

where \( \text{osc} \) is over \((t_{j-1}, t_{j}) \cap (t_{k-1}, t_{k})\) and \( \Delta g(t_{jk}) = g(\min\{t_{j}, t_{k}\}) - g(\max\{t_{j-1}, t_{k-1}\}) \). So

\[
\| \sum_{D} f \Delta x - \sum_{D'} f \Delta x \| < \epsilon
\]

and the limit exists.

**COROLLARY 5.5.** For \( f \in \text{BD}_{1}(T) \) and \( x \) of weak bounded variation on \( T, f \in \text{PS}(x) \) and

\[
\text{PS-} \int_{T} f(t) \, dx(t) = \text{MS-} \int_{T} f(t) \, dx(t)
\]
Proof. \( x^*[M^{*} \int_{T} f(t) \, dx(t)] = \int_{T} f(t) \, dx^*[x(t)]. \)

When the \( p \)-th moments of \( x(t) \) exist, \( t \in T \), the computation of the \( p \)-th moments of the PS-integral falls out from the definition. Let \( X = L_{1}(\Omega) \), then

\[
\mathbb{E}X^{A} = \int_{A} f(t) \, d\mathbb{E}x(t).
\]

When \( X = L_{2}(\Omega) \), take \( \xi^{*} = x_{A} \in L_{2}(\Omega) \) and

\[
\mathbb{E}X^{2}_{A} = \int_{A} f(t) \, \mathbb{E}[dx(t) \, x_{A}]
\]

\[
= \int_{A} \left[ \int_{A} f(s) \, f(t) \, \mathbb{E}[dx(s) \, dx(t)] \right]
\]

where we assume \( \Gamma(\cdot, \cdot) \in BV(T^{2}) \), \( \Gamma(s, t) = \mathbb{E}[x(s) \, x(t)] \). If \( x \) has orthogonal increments

\[
\mathbb{E}X^{2}_{A} = \int_{A} |f(t)|^{2} \, \mathbb{E}|dx(t)|^{2}.
\]

In general, if \( X = L_{p}(\Omega) \), \( 1 \leq p < \infty \), take \( \xi^{*} = x_{A}^{p-1} \in L_{q}(\Omega) \), \( q = p/(p-1) \), since \( \mathbb{E}|x_{A}^{p-1}|^{q} = \mathbb{E}|x_{A}|^{p} < \infty \) and

\[
\mathbb{E}X^{p}_{A} = \int_{A} f(s) \, \mathbb{E}[dx(s)] \left[ \int_{A} f(t) \, dx(t) \right]^{p-1}.
\]

For \( p \) an integer,

\[
\mathbb{E}X^{p}_{A} = \int_{A} \cdots \int_{A} f(s_{1}) \cdots f(s_{p}) \, \mathbb{E}[dx(s_{1}) \cdots dx(s_{p})],
\]

again assuming \( \Gamma \in BV(T^{p}) \), \( \Gamma(s_{1}, \ldots, s_{p}) = \mathbb{E}[x(s_{1}) \cdots x(s_{p})] \).
6. EXAMPLES

6.1. To illustrate Theorem 4.4, the representation for the PS-integral, consider a Poisson process on $T = [0, b]$ with parameter $\lambda > 0$. $V_T(x)$ is finite, hence the $\text{PS}$-integral may be defined (see below).

For this process, almost all sample paths are increasing step functions, integer-valued with jumps of magnitude one and continuous from the left. Also, there are only a finite number of discontinuities in any finite interval.

Let $f$ be continuous on $T$. Then the SP-integral exists and is

$$\int_T f(t) \, dx(t) = \sum_{k=1}^N f(d_k) \left[ x(d_k^+) - x(d_k^-) \right] = \sum_{k=1}^N f(d_k)$$

where $N = N_T$ is a random variable representing the number of discontinuities, $\{d_k(\cdot)\}$, of the sample path functions on $T$. So the (stochastic) integral of $f$ with respect to the Poisson process is a random sum of random variables.

For $D = \{t_j\}_{j=1}^m$ such that $\max_{1 \leq j \leq m} \Delta t_j < \max_{1 \leq k \leq N(\omega)} \Delta d_k(\omega)$ and $D \supset \{d_k(\omega)\}_{k=1}^{N(\omega)}$,

$$\sum_{j=1}^m x(t_j', \omega) \Delta m_f(t_j) = \sum_{j=2}^m (j-1) \{ \xi'(f(t_{j-1}) - f(t_j)) \}$$

where $t_j' \in (t_{j-1}, t_j)$ and $\xi'$ is the sum of differences $\Delta m_f$ on each subinterval $\Delta d_k(\omega)$. But $x(\cdot, \omega)$ is constant on these subintervals, so the telescoping $\xi'$ reduces to $f[d_{k-1}(\omega)] - f[d_k(\omega)]$ and

$$\sum_{j=1}^m x(t_j', \omega) \Delta m_f(t_j) = \sum_{k=1}^{N(\omega)} \xi f[d_k(\omega)] = \sum_{k=1}^{N(\omega)} f(d_k)$$

that is, $\text{PS-} \int_T f \, dx = \text{SP-} \int_T x \, dm_f = \sum_{k=1}^{N(\omega)} f(d_k)$.
Here, the SP-, PS- and ID- integrals all exist and coincide.

6.2. Define a process \( x_\gamma \) on \( T = [0, b] \) by letting the increments \( \Delta x_\gamma(t) \) be independent and normally distributed with mean \( \gamma \sigma \Delta t \) and variance \( \sigma^2 \Delta t ; \gamma, \sigma > 0 \). \( x_\gamma \) is a shift of the Brownian motion for \( \gamma > 0 \) and \( V_T(x_\gamma) \) is finite.

Here the PS- and ID- integrals may be defined, but not the SP-integral.

Using the above three examples and others, a table may be set up displaying all possible combinations of existence for the sample path, Pettis-Stieltjes and Ito-Doob integrals.

**REMARK.** The Brownian motion shift is an example of a process which induces a measure of finite variation on \( \mathbb{B}(T) \) (countably additive) but not of finite strong variation.

\[
E|\Delta x_1(t)| = \left[ \sqrt{2\Delta t} \int_0^{\sqrt{\Delta t/2}} e^{-\xi^2} d\xi + e^{-\Delta t/2} \right] E|\Delta x(t)|
\]

where \( x \) is (zero-mean) Brownian motion. Hence,

\[
E|\Delta x_1(t)| \geq E|\Delta x(t)| = \sigma \sqrt{2\Delta t / \pi}
\]

and

\[
S_T(x_1) \geq S_T(x) \geq \sigma \sqrt{2 / \pi} \sum_{k=1}^{n} \sqrt{\Delta t_{kn}} \rightarrow \infty .
\]
Recalling that $W_T(\mu) = \beta V_T(\mu)$ where $\mu: (T, Q) \to X$ and $1 \leq \beta \leq 2$, the Poisson process provides an example for which the equality is obtained ($\beta = 1$); let $\|x^*\| \leq 1$,

$$W_T(x, x^*) \leq \sup_{k=1}^{n} \mathbb{E}[\Delta x(t_{kn})] = \lambda b .$$

But

$$V_T(x) = \sup \mathbb{E} \left[ \sum_{k=1}^{n} [x(t_{kn}) - x(s_{kn})] \right]$$

$$= EN_T = \mathbb{E}x(b) = \lambda b ,$$

where $N_T$ is defined in 6.1. Therefore $W_T(x) \leq V_T(x)$, so $\beta = 1$. This may also be seen from the fact that

$$S_T(x) = \sup \mathbb{E}[\Delta x(t_{kn})] = \lambda b ,$$

so $V_T(x) = W_T(x) = S_T(x)$. In fact, these equalities obtain for any process in $L_1(\Omega)$ with nondecreasing (or nonincreasing) sample paths.

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APPENDIX

The properties of the scalar-valued modified Stieltjes integral are summarized here for reference. The integral is due to B. Dushnik and some results are listed in [3, p.273]. Proofs are omitted.

**DEFINITION A.1.** Let $f$ and $g$ be real-valued functions on $T$. The modified Stieltjes integral is defined as

$$\text{MS-} \int_T f(t) \, dg(t) = \lim_{D} \sum_{k=1}^{n} f(t_k') \Delta g(t_k)$$

where the limit is taken over successively finer partitions $D$ of $T$. The $t_k'$ are arbitrary interior points of the subintervals $(t_{k-1}, t_k)$.

Note that the MS-integral is more general than the usual Riemann-Stieltjes (RS-) integral.

**DEFINITION A.2.** $\text{osc}(f)$ is the oscillation of $f$ over a prescribed interval $T$ and equals $\sup \{ |f(s) - f(t)| : s, t \in T \}$.

**LEMMA A.3.** Let $f \in BD_1(T)$, then for every $\varepsilon > 0$, there exists a partition $D_\varepsilon$ of $T$ such that $\text{osc}(f) < \varepsilon$ over any open subinterval of $D_\varepsilon$ and, hence, over any open subinterval of $D \supset D_\varepsilon$.

**PROPOSITION A.4.** When $f \in BD_1(T)$ and $g \in BV(T)$, there exists

$$\text{MS-} \int_T f(t) \, dg(t) .$$

**PROPOSITION A.5.** Let $f \in BD_1(T)$ and $g$ continuous in $BV(T)$. Then there exists

$$\text{RS-} \int_T f(t) \, dg(t) = \text{MS-} \int_T f(t) \, dg(t).$$
PROPOSITION A.6. Let \( f \in \text{BD}_1(T) \) and \( g \in \text{BV}(T) \). Then

\[
\text{MS-} \int fdg = \text{RS-} \int fdg_c + \text{MS-} \int fdg_d
\]

\[
= \text{RS-} \int fdg_c + \sum_t \{f(t+)[g(t+)-g(t)] + f(t-)[g(t)-g(t-)]\}
\]

where \( g_c \) is the continuous part of \( g \) and \( g_d \) exhibits the discontinuities.

PROPOSITION A.7. For \( f \in \text{BD}_1(T) \) and \( g \in \text{BV}(T) \), the Lebesgue-Stieltjes (LS-) integral exists and

\[
\text{LS-} \int fdg = \text{MS-} \int fdg .
\]

PROPOSITION A.8. When both \( f \) and \( g \) are in \( \text{BV}(T) \), a type of integration by parts theorem holds.

\[
\int fdg + \int gdf = [gf]_a^b + \sum_x \{f(x-)[g(x)-g(x-)] - f(x)[g(x+)-g(x-)] + f(x+)[g(x+)-g(x)]\}
\]

where the sum is taken over the (common) discontinuities of \( f \) and \( g \).
REFERENCES


