CONSTRUCTION OF RULED SURFACES IN 5-DIMENSIONAL FINITE
PROJECTIVE GEOMETRIES

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1. Introduction. I want to describe, as briefly as possible, the methods for constructing one of my ruled surfaces in PG(5,q). Here q is any prime-power. However, the theory becomes trivial (and exceptional) for q=2. So assume

\[ q \geq 3 \quad (q \text{ a prime-power}). \]

2. Preliminary. Nothing is said in my paper on ruled surfaces (the Luxembourg paper) about reguli. I need the following as background:

**Lemma 2.1.** Let S be a surface in \( \Sigma = \text{PG}(5,q) \) ruled by 3 systems of planes, belonging, say, to classes I, II and III. Let L be a line lying in a ruling plane of (say) class I. Let \( R \) be the set of \( q+1 \) ruling planes of (say) class II each of which contains exactly one point of L. Then:

(i) \( R \) is a regulus of planes of \( \Sigma \).

(ii) Each of the \( q^2 \) and \( q+1 \) ruling planes of class I meets the \( q+1 \) planes of \( R \) in the \( q+1 \) points of a line -- a transversal line to \( R \).

(iii) Each of the \( q^2 \) and \( q+1 \) ruling planes of class III meets the \( q+1 \) planes of \( R \) in the \( q+1 \) points of a conic.

**Proof.** It is to be understood that S is constructed as in my Luxembourg paper. Set

* Lectures given at the University of North Carolina at Chapel Hill supported by the U.S. Air Force Office of Scientific Research under Grant No. AFOSR-68-1406.
\[ F = \text{GF}(q), \quad K = \text{GF}(q^3), \]

and let \( V = K \times K \) be the 6-dimensional vector space over \( F \) consisting of ordered pairs \((x,y), x, y \in K\), with

\[
\begin{align*}
(x,y) + (x',y') &= (x+x', y+y') \quad \forall \ x, y, x', y' \in K \\
f(x,y) &= (fx, fy) \quad \forall \ f \in F.
\end{align*}
\]

Then \( S \) consists of all points \(<(x,y)>\) such that

\[ N(x) = N(y); \]

equivalently

\[ x^{q^2+q+1} = y^{q^2+q+1}. \]

The classes (I), (II), (III) are defined as the set of planes of the following sorts:

\[
\begin{align*}
(I) \quad y &= kx \\
(II) \quad y &= kx^q \\
(III) \quad y &= kx^{q^2}
\end{align*}
\]

For any fixed \( k \) such that

\[ N(k) = k^{q^2+q+1} = 1. \]

For the proof, we may assume that \( L \) is the 2-dimensional vector space

\[ L = <(1,1), \quad (a,a)> \]

for some fixed \( a \),

\[ a \in K-F. \]

Then the \( q+1 \) points of \( L \) are the following: the point

\[ <(1,1)> \]
and \( q \) points of form

\[ <f(1,1) + (a,a) > = <(f+a, f+a)>, \quad f \in F. \]

The plane of class (II) through \( <(1,1)> \) is

\[ \Pi_{\infty} : y = x^q. \]

The plane of class (II) through \( <(f+a, f+a)> \) is

\[ \Pi_f : y = \frac{f+a}{f+a^q} x^q. \]

We may check that, for each \( k \in K, k \neq 0 \), the line

\[ L_k = <(k,k^q), (ka,k^q a)> \]

meets \( \Pi_{\infty} \) in \( <(k,k^q)> \) and \( \Pi_f \) in \( <(k(a+f),k^q(a^q+f)>, \) and hence is a transversal to

\[ R = \{\Pi_{\infty}\} \cup \{\Pi_f | f \in F\}. \]

Furthermore, \( L_k \) lies in the following plane of class I,

\[ y = k^{q-1} x. \]

We also note that

\[ L = L_k, \quad <\overrightarrow{k}, \frac{k'}{k} \in F, \]

so that the total number of distinct lines \( L_k \) is

\[ \frac{q^3 - 1}{q-1} = q^2 + q + 1. \]

Thus: \( R \) is a regulus of (skew) planes of \( \Sigma: PG(5,q) \), with the \( q^2+q+1 \) lines \( L_k \) as its transversal lines.
Next consider a typical ruling plane of class III, say

\[ a: \ y = kx^{q^2}, \]

where \( k \) is some fixed element of \( K \) with

\[ k^{q^2+q+1} = 1. \]

We can choose \( b \) (in \( q-1 \) ways) so that

\[ k = b^{q-1}. \]

Thus the equation of \( a \) becomes

\[ a: \ y = b^{q-1}x^{q^2}. \]

Then we may check that

\[ \Pi_0 \cap a = \langle b^{-q^2}, b^{-1} \rangle \]

\[ \Pi_f \cap a = \langle b^{-q^2}(f+a)^{-q^2}, b^{-1}(f+a)^{-1} \rangle, \quad \forall \ f \in F. \]

To complete the proof, we must show that the \( q+1 \) points so obtained lie on a conic (in \( a \)). If \( q \) is odd, it is enough to show that no three of the points lie on a line. Consider the special case of these points

\[ \Pi_{f_i} \cap a, \quad i = 1, 2, 3, \]

where \( f_1, f_2, f_3 \) are distinct elements of \( F \). If these points lie on a line, we must have
for elements \( u, v, w \) of \( F \), not all zero. Considering components, we get two equations for \( u, v, w \) which (after cancelling some non-zero factors) become

\[
\begin{align*}
(1) & \quad (f_1+a)^{-2} u + (f_2+a)^{-2} v + (f_3+a)^{-2} w = 0 \\
(2) & \quad (f_1+a)^{-1} u + (f_2+a)^{-1} v + (f_3+a)^{-1} w = 0.
\end{align*}
\]

We note that (2) implies (1). However, (2) (with \( u, v, w \) in \( F \), not all zero) means that \( a \) satisfies a quadratic equation with coefficients in \( F \). Since \( a \in K-F \), and \( K \) is three dimensional over \( F \), this is false.

Similarly, three points \( \Pi_0 \cap \alpha, \Pi_{f_1} \cap \alpha, \Pi_{f_2} \cap \alpha \), cannot be collinear.

We leave the proof of Lemma 2.1 at this point.

3. Construction of surfaces \( S \). (Some algebraic details.)

We suppose given a regulus \( R \) of planes of \( \Sigma = \text{PG}(5,q) \), consisting of \( q+1 \) skew planes \( \Pi_i \);

\[ R = \{ \Pi_i | i = 1, 2, \ldots, q+1 \} \]

such that each of the \( 2^q + q+1 \) transversal lines to \( \Pi_1, \Pi_2, \Pi_3 \) meets every \( \Pi_i \) in a point. We also suppose given one more plane, \( \Pi \), disjoint from the \( q+1 \) planes \( \Pi_i \).
We consider the problem of constructing a triply-ruled surface $S$ of $\Sigma$ such that the set

$$R \cup \{\Pi\}$$

of $q+2$ skew planes forms part of one of the three systems of $q^2+q+1$ ruling planes of $S$.

In the light of Lemma 2.1, we may assume that the planes of $R \cup \{\Pi\}$ belong to Class II, and that every plane of Class I contains a (unique) transversal line to $R$.

We may suppose that $\Sigma = PG(5,q)$ is given by a six-dimensional vector space $V$ over $F = GF(q)$ with a basis $t_1, t_2, t_3, t'_1, t'_2, t'_3$ chosen so that

- $\Pi_1 = J(\infty)$ has basis $t_1, t_2, t_3$
- $\Pi_2 = J(0)$ has basis $t'_1, t'_2, t'_3$
- $\Pi_3 = J(I)$ has basis $t_1 + t'_1, t_2 + t'_2, t_3 + t'_3$.

Then every plane skew to $\Pi_1 = J(\infty)$ has form $J(X)$ for a unique $3 \times 3$ matrix $X = (x_{ij})$ over $F=GF(q)$, where

$$J(X) \text{ has basis } x_{11}t_1 + x_{12}t_2 + x_{13}t_3 + t'_1,$n $$x_{21}t_1 + x_{22}t_2 + x_{23}t_3 + t'_2,$n $$x_{31}t_1 + x_{32}t_2 + x_{33}t_3 + t'_3.$$n

In particular, $R$ consists of $J(\infty)$ and the $J(fI)$, $f \in F$, where

$$fI = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}.$$
Also

\[ \Pi = J(U) \]

for some irreducible 3x3 matrix \( U \).

Every vector \( a \) in \( J(\infty) \) has form

\[ a = a_1 \ell_1 + a_2 \ell_2 + a_3 \ell_3 \]

for unique elements \( a_1, a_2, a_3 \) in \( F \). We define

\[ a^X = (a_1 x_{11} + a_2 x_{21} + a_3 x_{31}) \ell_1 + (a_1 x_{12} + a_2 x_{22} + a_3 x_{32}) \ell_2 + (a_1 x_{13} + a_2 x_{23} + a_3 x_{33}) \ell_3 \]

for every 3x3 matrix \( X \). In this notation

\[ J(X) \text{ has basis } \ell_1^X, \ell_2^X, \ell_3^X. \]

Let \( a \) be a non-zero vector in \( J(\infty) \), so that \( \langle a \rangle \) is a point of the plane \( J(\infty) \). The transversal line, \( L_a \), to \( \mathbb{R} \) through \( \langle a \rangle \) is the two-dimensional vector space

\[ L_a = \langle a, a' \rangle \]

where we define

\[ a' = a_1 \ell_1' + a_2 \ell_2' + a_3 \ell_3'. \]

A ruling plane, of our proposed surface \( S \), which has Class (I) and contains \( L_a \) must (in particular) meet \( \Pi = J(U) \) in a point, say in

\[ \langle b^U + b' \rangle, \]

where \( b \) is some non-zero vector in \( J(\infty) \). Then this ruling plane is (say)

\[ \alpha = \langle a, a', b^U + b' \rangle. \]
The ruling plane $\alpha$ should meet each plane of $R$ in a point, namely in a point of $L_a$. This puts some restrictions on the point $<b^U+b'>$ of $\Pi$ or, equivalently, the point $<b>$ of $J(\infty)$. To see this, we note that each point of $\alpha$ has form $<v>$ where $v$ is a non-zero vector of form

$$v = fa + ga' + h(b^U+b'),$$

where $f, g, h$ are elements of $F$, not all zero. Equivalently,

$$v = (fahb^U) + (gahb).$$

The point $<v>$ will be in $J(\infty)$ if and only if

$$ga + hb = 0.$$

We wish this condition to imply $h = 0$. Thus we want

(1) \quad $<$b$>$ \neq <0$>.

If $t \in F$, so that $J(tI)$ is in $R$, the point $<v>$ will be in $J(tI)$ if and only if

$$fa + h^U = t(gahb)$$

or

$$h(b^U - tb) = (-f+tg)a$$

or

$$h \cdot b^U - tI = (-f+tg)a.$$

We want this equation to imply $h=0$. Thus we want

$$<b^U - tI> \neq <a>$$

or, equivalently,

(2) \quad $<$b$>$ \neq $<a(U-tI)^{-1}>$, \quad \forall \ t \in F.$
It may be shown that the point \( \langle a \rangle \) of \( J(\infty) \) together with the \( q \) points
\[
\langle a(U-tI)^{-1} \rangle, \quad t \in F
\]
of \( J(\infty) \) form a conic of \( J(\infty) \). Hence, also, the corresponding points
of \( \Pi \), namely
\[
\langle aU+a' \rangle \text{ and } \langle (U-tI)^{-1}U + (a(U-tI)^{-1})' \rangle, \quad t \in F,
\]
form a conic of \( \Pi \). Also the point \( \langle bU+b' \rangle \) must avoid the \( q+1 \) points of the latter conic. This gives
\[
(q^2+q+1) - (q+1) = q^2
\]
choices of the point \( \langle bU+b' \rangle \) so that
\[
a = \langle a, a', bU+b' \rangle
\]
meets each of the \( q+2 \) planes in \( RU(\Pi) \) in (exactly) a point.

It is easy to check that the conditions on \( b \) are equivalent to the following:

(3) \( a, b, bU \) form a basis of \( J(\infty) \) over \( F = GF(q) \).

However, since \( U \) is irreducible and \( b \neq 0 \), the vectors \( b, bU, bU^2 \)
form a basis of \( J(\infty) \) over \( F = GF(q) \). Hence (3) means that
\[
a = fb + gbU + hbU^2
\]
for some \( f, g, h \) in \( F \) with \( h \neq 0 \). Equivalently (the case \( h=1 \))

(3') \( \langle a \rangle = \langle f_0 b + g_0 bU + bU^2 \rangle \)
for some \( f_0, g_0 \) in \( F \). (Note that the ordered pair \( f_0, g_0 \) can be chosen in \( q^2 \) ways.)

Assuming (3'), we can write the plane \( \alpha \) as the plane \( \alpha(b) \) where

\[
\alpha(b) = \langle f_0 b + g_0 b^U, f_0 b' + g_0 b^U', f_0 b'' + g_0 b^U'' \rangle.
\]

Here \( b \) can be any non-zero vector in \( J(\infty) \). If, now, for fixed \( f_0, g_0 \) in \( F \), we consider all the planes \( \alpha(b) \), we check easily that they form a set of

\[
q^2 + q + 1
\]
distinct, mutually skew, planes, each of which meets every plane of \( R \cup \Pi \) in exactly a point, and each of which contains a unique transversal line to \( R \), namely the line

\[
\langle f_0 b + g_0 b^U, f_0 b' + g_0 b^U', f_0 b'' + g_0 b^U'' \rangle.
\]

Although we are not supplying a proof here, these \( q^2 + q + 1 \) planes \( \alpha(b) \) constitute the desired (complete) collection of planes of Class I.

We still have to supply the

\[
q^2 + q + 1 - (q+2) = q^2 - 1
\]

missing planes of Class II, and the \( q^2 + q + 1 \) planes of Class III.

4. Geometric approach. The material of Section 3 may be summarized as follows:

Suppose given a regulus, \( R \), of \( q+1 \) skew planes of \( E = PG(5,q) \) and a plane \( \Pi \) disjoint from each of the planes in \( R \). Then, in precisely \( q^2 \) distinct ways, we can set up a one-to-one correspondence

\[
(4.1) \quad L \leftrightarrow P
\]
between the $q^2+q+1$ transversal lines $L$ to $R$ are the $q^2+q+1$ points $P$ of $\Pi$ such that:

(i) If $L$, $P$ are a corresponding pair, the plane $L+P$ intersects $\Pi$ in $P$ and intersects each plane $\Pi_1$ of $R$ in the point $L\cap\Pi_1$.

(ii) The $q^2+q+1$ planes $L+P$ obtained by letting $L$, $P$ range over all corresponding pairs are structurally skew.

This is essentially all we need to complete the construction of the surface $S$.

We assume that some fixed correspondence (4.1) has been chosen. Consider any corresponding pair $L$, $P$. We note first that the projective space

$$
H_P = L + \Pi
$$

is four-dimensional and hence is a hyperplane of $\Sigma = PG(5,q)$. Next we construct a projective 3-space,

$$
T_P
$$
in the following manner: Let $\Pi_1$, $\Pi_2$ be two distinct planes of $R$ and let $M$ be the unique transversal line through $P$ to $\Pi_1$, $\Pi_2$, meeting $\Pi_1$, $\Pi_2$ in points $P_1$, $P_2$ respectively. Let $L_1$, $L_2$ be the transversal lines to $R$ through $P_1$, $P_2$ respectively. Note that $L_1 \neq L_2$, else the point $P$ of $\Pi$ would lie on a transversal to $R$ and hence on one of the planes of $R$, a contradiction. Since $L_1 \neq L_2$,

$$
T_P = L_1 + L_2
$$
is a projective 3-space. Since $M$ contains $P$, $P_1$, $P_2$ and since $P_1$ is on $L_1$, $P$, is on $L_1$, then $T_P$ contains $P$ and $M$ is the transversal through $P$ to $L_1$, $L_2$ in $T_P$. 
We need to note the following:

(a) The 3-space \( T_p \) depends only on \( R \) and \( P \), not on \( \Pi \) or on the choice of the planes \( \Pi_1, \Pi_2 \) of \( R \).

(b) \( T_p \) meets each plane of \( R \) in a line, giving \( q+1 \) such lines, and contains precisely \( q+1 \) transversals to \( R \), namely the transversals to the first set of lines. These two sets of \( q+1 \) lines form the two sets of rulings of a doubly-ruled quadric

\[
Q_p
\]
of \( T_p \); and \( Q_p \) depends only on \( P \) and \( R \).

(c) \( P \) lies on exactly \( q+1 \) planes to \( Q_p \) in \( T_p \). Each of these contains exactly one transversal line to \( R \) and meets exactly one plane of \( R \) in a line. Each of the remaining \( q^2 \) planes of \( T_p \) through \( P \) meets \( Q_p \) in a conic, contains no transversal line to \( R \), and meets no plane of \( R \) in a line.

(d) \( T_p \cap \Pi = P \).

Next we need the following:

(4.4) \( T_p \cap L \) is the empty point-set.

(4.5) \( \Pi_p = T_p \cap H_p \) is a plane (disjoint from \( L \)).

To prove (4.4), first suppose that \( L \) is contained in \( T_p \). Then \( P+L \) is a tangent plane to \( Q_p \) and hence meets some plane of \( R \) in a line. This is a contradiction. Hence \( L \) is not in \( T_p \). Next suppose that \( L \) meets \( T_p \) in a point \( P' \). (Since \( P \) is not on \( L \), necessarily \( P' \neq P \).) Since \( P' \) is on \( L \), then \( P' \) is on some plane, say \( \Pi_i \), of \( R \). Since \( P' \) is in \( \Pi_i \cap T_p \), the \( P' \) is on \( Q_p \). In particular, the transversal line through \( P' \) to \( R \) is a ruling of \( Q_p \), and is in \( T_p \). But this transversal, being the unique transversal line through \( P' \) to
R, must be L. Hence L is in $T_p$, a contradiction. Therefore, (4.4) must be true.

In view of (4.4), the projective space $T_p + L$ has dimension $3 + 1 - (-1) = 5$. Therefore,

\[(4.6) \quad T_p + L = \sum.\]

Since $H_p$ contains L, we see from (4.6) that

\[T_p + H_p = \sum.\]

Hence

\[\dim(III_p) = \dim T_p + \dim H_p - \dim \sum = 3 + 4 - 5 = 2.\]

Thus $III_p$ is a plane. Moreover

\[III_p \cap L = T_p \cap H_p \cap L\]

is empty by (4.4). This proves (4.5). From (4.3) we get

\[(4.7) \quad III_p + L = H_p.\]

Indeed, the left-hand side of (4.7) is contained in the right-hand side, and both sides have projective dimension 4. From (4.7) and the fact that $P$ is in $III_p$ we get

\[III_p + (P+L) = H_p\]

and hence

\[(4.8) \quad III_p \cap (P+L) = P.\]
Next we need

(4.9) If $\Sigma_4$ is a projective 4-space of $\Sigma$ containing $\Pi$, then $\Sigma_4$ contains one and only one transversal line to $R$.

To see this, we note that there are precisely $q^2+q+1$ distinct transversal lines to $R$ and precisely $q^2+q+1$ distinct projective 4-spaces of $\Sigma$ containing $\Pi$. If (4.9) is false, there must be a $\Sigma_4$ which contains $\Pi$ but contains no transversal line to $R$. Let $\Sigma_4$ be such a 4-space, and let $\Pi_1$ be one of the planes of $R$. Then $\Sigma_4$ intersects $\Pi_1$ is a line, say $M$. The $q+1$ transversal lines to $R$ through the points of $M$, each meet $\Sigma_4$ in a single point, namely a point of $M_1$. The same process, carried out for the $q+1$ distinct planes of $R$, shows that there must be at least

$$(q+1)^2 > q^2+q+1$$

distinct transversal lines to $R$. This is a contradiction. Hence (4.9) is true. As a special case of (4.9),

(4.10) $L$ is the only transversal line to $R$ contained in $H = L + \Pi$.

By (4.10) and (4.4), (4.5),

(4.11) The plane $\Pi_p$ contains no transversal line to $R$.

In view of (4.11), $\Pi_p$ meets the $q+1$ planes of $R$ in the points of a conic. (The conic lies on $Q_p$.) The $q^2+q+1$ planes $\Pi_p$, one for each point $P$ of $\Pi$, are our candidates for the ruling planes of Class III. (Compare Lemma 2.1.) We omit the proof that every two of these are disjoint.
Of course, the planes of class I are the planes

\[ I_P = P + L \]

where \( L, P \) is a corresponding pair in the sense of (4.1). It has to be shown that every plane of class I meets every plane of class III in a point.

In addition, we have in class II only the set

\[ R \cup \{\Pi\} \]

of \( q+2 \) planes. But it should be clear at this point that the missing \( q^2-1 \) planes are uniquely determined by the planes of Classes I and III. We have only to consider Lemma 2.1 with classes I, II, III replaced (for example) by classes III, I, II respectively.

I will stop here. Note that one must prove that the construction can actually be completed consistently. (This is, in fact, true.)