ON FIXED SIZE CONFIDENCE BANDS FOR THE BUNDLE STRENGTH OF FILAMENTS

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Institute of Statistics Mimeo Series No. 633

June, 1969
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Summary. The distribution theory of the bundle strength of filaments (for fixed sample sizes) has been studied by Daniels (1945) and Bhattacharyya et al. (1969). The object of the present investigation is to develop, along the lines of Anscombe (1952) and Chow and Robbins (1965), the asymptotic theory of fixed-width (sequential) confidence bands for the bundle strength of filaments. In this context, some convergence results on the empirical distribution and on the bundle strength are also established.

1. Statement of the problem. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) non-negative valued random variables with an absolutely continuous cumulative distribution function (cdf) $F(x)$, defined on $(0, \infty)$. We assume that

\begin{equation}
\lambda^2 = \int_0^\infty x^2 dF(x) < \infty \quad \text{and} \quad \theta = \sup_x x[1-F(x)] = x_o[1-F(x_o)] > 0,
\end{equation}

where $x_o$ is the unique point where the maximum is attained. Note that

\begin{equation}
0 < \int_0^\infty xdF(x) < \lambda, \quad 0 < x_o < \infty \quad \text{and} \quad 0 < \theta = F(x_o) < 1.
\end{equation}

Also, by assumption, the derivative of $x[1-F(x)]$ vanishes at $x_o$, and hence, $f(x_o) = F'(x_o) > 0$. Further, we assume that for $\delta(>0)$, sufficiently small,

\begin{equation}
x[1-F(x)] \leq 0 - C|x-x_o|^k \quad \text{for all} \quad |x-x_o| < \delta,
\end{equation}

\textsuperscript{1) Work supported by the Army Research Office, Durham, Grant DA-ARO-D-31-124-G746.}
where $C(>0)$ and $k(>1)$ are suitable constants. In fact, if $x[1-F(x)]$ is twice differentiable at $x_0$, then (1.3) holds with $k=2$.

Our parameter of interest is $\theta$, where $\theta \in (0, \lambda)$. For a random sample $X_1, \ldots, X_n$ of size $n$ from the distribution $F(x)$, we denote the ordered random variables by $X_{n,1} \leq \ldots \leq X_{n,n}$. Let then

\[(1.4) \quad D_n = \max_{1 \leq i \leq n} [ (n-i+1)X_{n,i} ] = (n-r+1)X_{n,r_n}, \quad \text{and} \quad Z_n = n^{-1}D_n;\]

by virtue of (1.1) and the assumed continuity of $F(x)$, $r_n (1 \leq r_n \leq n)$ is unique, with probability 1. It is shown in [3] that $Z_n$ almost surely converges to $\theta$.

When the $X_i$ represent the breaking stresses of filaments, $D_n$ is equal to the maximum stress which a bundle of $n$ filaments can stand and is termed the bundle strength (cf. [3,5]). We term $\theta$ as the mean (per unit) bundle strength of filaments. We want to find a confidence interval for $\theta$ of prescribed width $2d$ and prescribed confidence coefficient $\gamma (0 < \gamma < 1)$. Since, neither $F(x)$ nor the distribution of $D_n$ is explicitly known, no fixed sample procedure sounds available.

For $n \geq 1$, we define

\[(1.5) \quad I_n(d) = [Z_n - d, Z_n + d], \quad d > 0 \text{ and } p_n = (n+1)^{-1}r_n.\]

Also, let $\{a_n\}$ be a sequence of known constants such that

\[(1.6) \quad \lim_{n \to \infty} a_n = a \quad \text{where} \quad (2\pi)^{-\frac{1}{2}} \int_{-a}^{a} \exp(-\frac{1}{2}t^2) dt = \gamma.\]

Finally, let

\[(1.7) \quad v_n(d) = a^2 z^2 p_n /[d^2 (1-p_n)], \quad v(d) = a^2 \theta^2 \pi \theta /[d^2 (1-\theta)].\]

In order to estimate $\theta$ with small dispersion $d$, we consider the following sequential procedure: continue sampling until first $v_n(d) \leq n$, and then compute $Z_n$ and $I_n(d)$. 
Theorem 1.2 (to follow) establishes the asymptotic properties (viz., consistency and efficiency) of this procedure. Now, for the stopping (random) variable $N(d)$ defined by

$$N(d) = \text{smallest integer } n(\geq 1) \text{ for which } v_n(d) \leq n,$$

we want to prove the following theorems.

Theorem 1.1. (i) If $\lambda < \infty$, then for every $d > 0$, the sequential procedure terminates with probability 1 and $E\left[\left(N(d)\right)^s\right] < \infty$ for every finite $s(>0)$. (ii) If $M(t) = E(e^{tx}) < \infty$ for some $t > 0$, then for every $d > 0$, $E(e^{tN(d)}) < \infty$ when $t \in (-\infty, t_0)$, where $t_0 > 0$.

Theorem 1.2. Under (1.1) through (1.3),

$$\lim_{d \to 0} \frac{[N(d)/v(d)]}{d} = 1 \text{ a.s.}, \quad \lim_{d \to 0} P\{\theta \in I_N(d)\} = \gamma,$$

$$\lim_{d \to 0} \{E[N(d)]/v(d)\} = 1.$$

The proofs of these theorems are postponed to Section 4.

It may be noted that the estimate $Z_n$ of $\theta$ or $v_n(d)$ of $v(d)$ is not linear in the $X_i$ or $X_i^2$ etc., and hence, the results of Anscombe (1952) or of Chow and Robbins (1965) are not directly applicable. In fact, we need to prove some almost sure convergence results on $\{Z_n\}$ (see Section 3) for the validation of the condition of "uniform continuity in probability" of Anscombe (1952) (and also implicit in [4]).

This, in turn, requires certain convergence results on the empirical cdf which are studied in Section 2. The last section develops (sequential) fixed percentage error confidence bands for $\theta$.

2. Some results on the empirical cdf. Define the empirical cdf $F_n(x)$ by

$$F_n(x) = n^{-1} \sum_{i=1}^{n} c(x-X_i), \quad 0 \leq x < \infty.$$
Theorem 2.1. If $\lambda<\infty$, given $\varepsilon>0$, for every $s(>0)$, we can find a positive $C_s$ and a $n_0(\varepsilon,s)$, such that for all $n\geq n_0(\varepsilon,s)$,

\begin{equation}
P\left\{ \sup_{0<x<\infty}|x[F_n(x)-F(x)]| > \varepsilon \right\} \leq C_s n^{-s}.
\end{equation}

Proof. It suffices to prove (2.2) only for some positive integer $s$. Consider a double sequence of real numbers

\begin{equation}
b_j(n) = j \cdot n^{-1}, \quad j=1, \ldots, n^* \quad \text{where} \quad n^* \sim n^{(s+3)/2},
\end{equation}

and define the random variables

\begin{equation}
U_{nj} = b_j(n) \left[F_n(b_j(n))-F(b_j(n))\right], \quad j=1, \ldots, n^*, \quad U_{n0} = 0.
\end{equation}

Now, for any positive integer $k$, we have for $1\leq j \leq n^*$,

\begin{equation}
E[U_{nj}^{2k}] = [b_j(n)]^{2k}E\left[[F_n(b_j(n))-F(b_j(n))]^{2k}\right]
\end{equation}

\begin{align}
&\leq n^{-k}[b_j(n)]^{2k}m_k[F(b_j(n))[1-F(b_j(n))]^{k} \text{ (where } m_k<\infty) \\
&\leq n^{-k}m_k^{2k} = n^{-k}m_k^2; \quad m_k<\infty,
\end{align}

as $\sup_j [b_j(n)]^{2k}[1-F(b_j(n))] \leq \sup_x x^{2k}[1-F(x)]<\lambda^2$. Thus, by the Markov inequality, for every $\varepsilon_1>0$,

\begin{equation}
P\{ |U_{nj}| > \varepsilon_1 \} \leq n^{-k}(m_k^2/\varepsilon_1^{2k}), \quad \text{for } j=1, \ldots, n^*,
\end{equation}

and hence, by the Bonferroni inequality

\begin{equation}
P\left\{ \sup_{1\leq j \leq n^*} |U_{nj}| > \varepsilon_1 \right\} \leq n^*n^{-k}(m_k^2/\varepsilon_1^{2k}) = n^{-k-\frac{s}{2}}(m_k^2/\varepsilon_1^{2k}).
\end{equation}

Now, for $b_j^{(n)}<x<b_{j+1}^{(n)}$, we have for $0\leq j<n^*-1$

\begin{equation}
|F_n(x)-F(x)| \leq \max(U_{nj},U_{nj+1})+b_j^{(n)}[F(b_{j+1}^{(n)})-F(b_j^{(n)})].
\end{equation}
When \( j = 0 \), the second term on the r.h.s. of (2.8) is equal to \( b_1^{(n)}[F(b_1^{(n)})] \), and it converges to zero (as \( n \to \infty \)) because \( b_1^{(n)} = n^{-1} \to 0 \) as \( n \to \infty \). Also, for \( j \geq 1 \), \( b_{j+1}^{(n)}/b_j^{(n)} < 2 \) and \( b_{j+1}^{(n)} - b_j^{(n)} = n^{-1} \to 0 \) as \( n \to \infty \). Hence,

\[
(2.9) \quad b_{j+1}^{(n)}[F(b_{j+1}^{(n)}) - F(b_j^{(n)})] \leq 2b_j^{(n)}[F(b_{j+1}^{(n)}) - F(b_j^{(n)})] \quad \leq 2 \int_{b_j^{(n)}}^{b_{j+1}^{(n)}} x dF(x) \to 0 \text{ as } n \to \infty, \text{ for all } j = 1, \ldots, n^*-1.
\]

Consequently, we can make the l.h.s. of (2.9) less than any preassigned small \( \eta > 0 \) when \( n \) is sufficiently large. Hence, from (2.7), (2.8) and (2.9), we have for \( n \) sufficiently large

\[
(2.10) \quad P\left( \sup_{0 \leq x < b_{n^*}^{(n)}} |x[F_n(x) - F(x)]| > \varepsilon \right) < n^{-(k-1/2)(s+3)}(m^*/\varepsilon^2k),
\]

where \( \varepsilon < \eta \), \( \eta = \varepsilon - \varepsilon \downarrow 0 \). Further, upon using the fact that \( b_{n^*}^{(n)} \sim n^{(s+1)/2} \) and by \( \lambda = \infty \) that \( [b_{n^*}^{(n)}]^2[1-F(b_{n^*}^{(n)})] \to 0 \) as \( n \to \infty \), we have

\[
(2.11) \quad P\{ \text{at least one } X_i > b_{n^*}^{(n)} \} = 1 - [F(b_{n^*}^{(n)})]^n
\]

\[
= 1 - [1 - [1 - F(b_{n^*}^{(n)})]^n] = n[1 - F(b_{n^*}^{(n)})] + O[(n[1 - F(b_{n^*}^{(n)})])^2]
\]

\[
= n\cdot o(n^{-s-1}) = o(n^{-s}).
\]

From (2.10) and (2.11), the result directly follows after noting that

\[
(2.12) \quad P\left( \sup_{0 \leq x \leq b_{n^*}^{(n)}} |x[F_n(x) - F(x)]| > \varepsilon \right)
\]

\[
\leq P\left( \sup_{0 \leq x < b_{n^*}^{(n)}} |x[F_n(x) - F(x)]| > \varepsilon \right) + P\{ \text{at least one } X_i > b_{n^*}^{(n)} \}
\]

\[
\leq C_s n^{-s} + o(n^{-s}),
\]
where we put in (2.10), $k > 3(s+1)/2$. Q.E.D.

Remark. The theorem readily extends to the case where $F(x)$ is defined on $(-\infty, \infty)$ and we consider a function $g(x)$ such that $\int_{-\infty}^{\infty} g^2(x) dF(x) < \infty$. In this case, for sufficiently large $n$, for every $\varepsilon > 0$ and $s > 0$,

\begin{equation}
(2.13) \quad P\{ \sup_{-\infty < x < \infty} |g(x)[F_n(x) - F(x)]| > \varepsilon \} \leq C_s n^{-s}.
\end{equation}

Also, the proof extends to the case when the $X_i$ can have possibly different cdfs $F_1, F_2, \ldots$, provided we replace $F(x)$ by $\bar{F} = n^{-1} \sum_{i=1}^{n} F_i$ and assume that

$$\sup_{n \geq 1} \int_{-\infty}^{\infty} g^2(x) d\bar{F}_n(x) < \infty.$$ 

Theorem 2.1 shows that the rate of convergence is faster than any power of $n$. It is naturally tempting to show that this can as well be an exponential rate. However, this requires a more stringent condition on $F(x)$, and is stated below.

Theorem 2.2. If $M(t) = E(e^{tx}) < \infty$ for some $t > 0$, then for every $\varepsilon > 0$, there exists a positive $\rho(\varepsilon)(<1)$, such that for sufficiently large $n$,

\begin{equation}
(2.14) \quad P\{ \sup_{0 < x < \infty} |x[F_n(x) - F(x)]| > \varepsilon \} \leq [\rho(\varepsilon)]^n.
\end{equation}

Proof. We define $\xi$ by $F(\xi) = p_o$ where $1/2 < p_o < 1$. If instead of (2.5) and (2.6), we use theorem 1 of Hoeffding (1963) [namely, his (2.3)], (where note that for all $x < \xi$, $x^2 < \xi^2$ and $x^2 F(x)[1-F(x)] \leq p_o \lambda^2$), we obtain as in (2.7) through (2.10) that for sufficiently large $n$

\begin{equation}
(2.15) \quad P\{ \sup_{0 < x < \xi} |x[F_n(x) - F(x)]| > \varepsilon \} \leq [\rho_1(\varepsilon)]^n; 0 < \rho_1(\varepsilon) < 1.
\end{equation}

for $x > \xi$, we let

\begin{equation}
(2.16) \quad b_j^{(n)} = \xi + jn^{-1}, j = 1, \ldots, n^*, n^* = \varepsilon \delta n, \delta > 0,
\end{equation}

where $\delta$ (may depend on $\varepsilon$) will be selected later on. Also, we define $u_{nj}$ as in (2.4). From theorem 1 of Hoeffding (1963) [namely (2.2)], it follows that
for all \( j > 1 \) (as \( x^2[1-F(x)] < \lambda^2 \) for all \( x \)). Also, from theorem 3 of Hoeffding (1963) [namely (2.8)], we obtain after some simplifications

\[
(2.17) \quad P_{\mathbf{u}_{nj} > \epsilon} \leq \exp\{-n\epsilon^2/(2x^2F(x)(1-F(x)))\} \leq \exp\{-n\epsilon^2/\lambda^2\}
\]

Now, \( M(t) \propto e^{tx}[1-F(x)] = 0 \) as \( x \to \infty \) and hence \( tx + \log[1-F(x)] \) is negative for \( x \) sufficiently large. Consequently, \( -x^{-1}\log[1-F(x)] \geq t > 0 \), for \( x \) large. Thus, if we select \( \xi \) such that \( e^{tx}[1-F(\xi)] < 1 \), we have after some manipulations that for sufficiently large \( n \)

\[
(2.19) \quad P_{\mathbf{u}_{nj} > \epsilon} \leq [\rho_3(\epsilon)]^n, \ \text{where} \ P_3(\epsilon) < 1.
\]

Hence, upon noting that \( \exp(-\epsilon^2/\lambda^2) \leq \rho_2(\epsilon) < 1 \), we have from (2.17) and (2.19) that

\[
(2.20) \quad P_{|\mathbf{u}_{nj}| > \epsilon} \leq [\rho_2(\epsilon)]^n + [\rho_3(\epsilon)]^n, \ \text{(1 \leq j \leq n*)}
\]

for \( n \) sufficiently large. Consequently, by the Bonferroni inequality

\[
(2.21) \quad P\left\{ \sup_{1 \leq j \leq n^*} |\mathbf{u}_{nj}| > \epsilon \right\} \leq n*\{[\rho_2(\epsilon)]^n + [\rho_3(\epsilon)]^n\}

\leq \{e^{\delta \rho_2(\epsilon)}\}^n + \{e^{\delta \rho_3(\epsilon)}\}^n.
\]

As both \( \rho_2(\epsilon) \) and \( \rho_3(\epsilon) \) are strictly less than unity, there exists a \( \delta(>0) \) such that \( e^{\delta \rho_i(\epsilon)} = \rho_i^*(\epsilon) < 1, \ i=2,3 \). Hence, proceeding as in (2.8) through (2.10), we have for \( n \) sufficiently large

\[
(2.22) \quad P\left\{ \sup_{\xi x < b(n)}|x[F(x)-F(\mathbf{u})]| > \epsilon \right\} \leq [\rho_2^*(\epsilon)]^n + [\rho_3^*(\epsilon)]^n.
\]

Also, note that \( M(t) \propto e^{\theta n^*} [1-F(b(n))] = 0 \) (as \( n \to \infty \)) \( \Rightarrow 1-F(b_n) = o(e^{-\theta n^*}) = o(e^{-n^{\delta^*} \cdot n^{-1}}) \). Hence, as in (2.11), we obtain that
At least one $x_i > b(n)$, where $\delta' < \delta$ and $\rho_4 = e^{-\delta'}. Hence, proceeding as in (2.12), we obtain from (2.15), (2.22) and (2.23) that for sufficiently large $n$

(2.24) $P\{\sup_{0 \leq x < \infty} |x[F_n(x) - F(x)]| > \epsilon\}
\leq 4\{\max[\rho_1(\epsilon), \rho_2(\epsilon), \rho_3(\epsilon), \rho_4(\epsilon)]\}^n = 4[\rho(\epsilon)]^n$. Q.E.D.

From theorem 2.1 it follows by letting $s > 1$ and using then the Borel-Cantelli lemma that for every $\epsilon > 0$, as $n \to \infty$

(2.25) $[\lambda < \infty] \Rightarrow \sup_{0 \leq x < \infty} |x[F_n(x) - F(x)]| < \epsilon$, with probability 1,

i.e., $\sup_{0 \leq x < \infty} |x[F_n(x) - F(x)]| = 0$ a.s. Also, based on the Kolmogorov bound, we have

(2.26) $\sup_{0 \leq x < \infty} |x[F_n(x) - F(x)]| = O(n^{-1/2})$, in probability,

for every finite $C$. Further, we state (without proof) that for $K > 1$

(2.27) $\sup_x |F_n(x) - F(x)| \leq K n^{-1/2}/\log n$, with probability 1, as $n \to \infty$,

and as a result, we have

(2.28) $\sup_{x < C} |x[F_n(x) - F(x)]| = O(n^{-1/2}/\log n)$, with probability 1, as $n \to \infty$,

for every finite $C$.

Let us now write

(2.29) $a_{n, \alpha} = n^{-\alpha} \log n, A(n, \alpha) = [x_o - a_{n, \alpha}, x_o + a_{n, \alpha}]$,

(2.30) $G_n(x_o, \alpha) = \sup [n^{1/2} |F_n(x) - F(x)| - [F_n(x_o) - F(x_o)] : x \in A(n, \alpha)]$. 

Theorem 2.3. For any $0 < \alpha < 1$, as $n \to \infty$

\begin{equation}
G_n(x_0, \alpha) = O(n^{-\alpha/2} \log n), \text{ with probability 1.}
\end{equation}

Proof. We consider $2n^B(\beta > \frac{1}{2})$ equi distant points on $A(n, \alpha)$ and apply the same trick as in lemma 1 of Bahadur (1966), who considered the special case $\alpha = \frac{1}{2}$; for brevity, the details are omitted. Further, if the distributions are not all same, we can proceed as in Sen (1968) and derive the same result (on replacing $F$ by $\overline{F}_n$).

Let us now define

\begin{equation}
M_n = n^k \{x_0[F_n(x_0) - F(x_0)]\} \text{ and } \zeta^2 = x_0^2 \pi_o (1 - \pi_o).
\end{equation}

Theorem 2.4. For every positive $\varepsilon$ and $\eta$, there exists a $\delta(>0)$, such that for all $n \geq n_0(\varepsilon, \eta)$ and $N \leq \delta n$

\begin{equation}
P\{\sup_{1 \leq j \leq N} |M_{n+j} - M_n| > \varepsilon \zeta\} < \eta.
\end{equation}

Proof. From (2.1) and (2.32), it follows that $M^*_n = n^k\overline{M}_n$ forms a martingale sequence, and hence, by the well-known Kolmogorov inequality

\begin{equation}
P\{\sup_{1 \leq j \leq N} |M^*_N - M^*_1| > t_1\} \leq t_1^{-2}E[M^*_N] \leq n(1 + \delta)(\zeta/t_1)^2,
\end{equation}

for all $N \leq \delta n$. Also, for all $1 \leq j \leq N \leq \delta n$,

\begin{equation}
|M_{n+j} - M_n| = |(n+j)^{-1/2}M^*_{n+j} - n^{-1/2}M^*_n| \leq n^{-1/2}|M^*_n| + [(1 + \delta)^{1/2} - 1]/(1 + \delta) \overline{M}_n \overline{M}_n.
\end{equation}

Hence, it follows from (2.34) and (2.35) that we are only to show that for every positive $\varepsilon'(\leq \varepsilon)$ and $\eta'(\leq \eta)$, there exists a $\delta(>0)$ such that

\begin{equation}
P\{\sup_{1 \leq j \leq N} |M^*_N - M^*_1| > \eta' \overline{\varepsilon}\} < \eta' \text{ for all } N \leq \delta n.
\end{equation}
Now, by the Kolmogorov inequality on martingales, we have

\[
(2.37) \quad \Pr\{ \sup_{1 \leq j \leq N} |M_{n+j}^* - M_j^*| > n^2t \} \leq \frac{(E(M_{n+j}^* - M_j^*)^2)}{(nt^2)}(\delta^2/t^2)
\]

Hence, (2.36) follows from (2.37) by letting \(\delta \leq \delta n(\varepsilon')^2\). Q.E.D.

3. Convergence of \(Z_n\) and \(p_n\). We shall study here the strong convergence of \(Z_n\) and \(p_n\) and also some exponential bounds for their convergence.

**Theorem 3.1.** If \(\lambda < \infty\), for every positive \(\varepsilon\) and \(s\), there exists a positive \(C_s\) and an \(n_0(\varepsilon, s)\), such that for all \(n > n_0(\varepsilon, s)\)

\[
(3.1) \quad \Pr\{ |Z_n - \theta| > \varepsilon \} \leq C_s n^{-s}.
\]

If \(M(t) = E(e^{tx}) < \infty\) for some \(t > 0\), then for sufficiently large \(n\),

\[
(3.2) \quad \Pr\{ |Z_n - \theta| > \varepsilon \} \leq [\rho(\varepsilon)]^n \quad \text{for some } \rho(\varepsilon) < 1.
\]

**Proof.** Corresponding to the empirical cdf \(F_n(x)\) in (2.1), we define

\[
(3.3) \quad F_n^*(x) = F_n(x - \theta) \quad \text{for all } x > 0.
\]

Then, by (1.4), we have

\[
(3.4) \quad Z_n = \sup_{x \in \mathbb{R}} x\left[1 - F_n^*(x)\right] - \sup_{x \in \mathbb{R}} x\left[1 - F_n(x)\right] = \sup_{x \in \mathbb{R}} x\left[1 - F_n^*(x)\right].
\]

Since \(F(x)\) is assumed to be absolutely continuous \(F(x - \theta) = F(x)\) and hence, theorems 2.1 and 2.2 hold good even if \(F_n(x)\) is replaced by \(F_n^*(x)\). Also, the events that \(\theta = x_0\left[1 - F(x_0)\right] = \sup_{x \in \mathbb{R}} x\left[1 - F(x)\right]\) and \(\left|x\left[F_n^*(x) - F(x)\right]\right| \leq \varepsilon\) imply that

\[
|Z_n - \theta| = \sup_{x \in \mathbb{R}} x\left[1 - F_n^*(x)\right] - \sup_{x \in \mathbb{R}} x\left[1 - F_n(x)\right] \leq \sup_{x \in \mathbb{R}} x\left|F_n^*(x) - F(x)\right| \leq \varepsilon.
\]

Hence, the proof directly follows from theorems 2.1 and 2.2. Q.E.D.
Note that by letting $s>1$ in (3.1) and using the Borel-Cantelli lemma, we obtain that

$$\lambda^{\infty} \Rightarrow Z_n = 0 \text{ a.s.} \quad (3.5)$$

For our purposes, we require some relatively more stronger results, stated below.

**Theorem 3.2.** Under (1.1) and (1.3), $[n^{1/2}(Z_n - \theta) + M_n] \rightarrow 0$, with probability one, as $n \rightarrow \infty$.

**Proof.** Since $x[1-F(x)]$ has a unique maximum $\theta$ at $x_0$, for every (small) $\varepsilon > 0$, there exists a $\delta(>0)$, such that

$$x[1-F(x)] \leq \theta - 2\varepsilon \text{ for all } |x-x_0|>\delta. \quad (3.6)$$

Thus, by (2.25), (3.3) and (3.6),

$$\sup\{x[1-F_n(x)] : |x-x_0|>\delta \} \leq \theta - \varepsilon, \text{ with probability 1, as } n \rightarrow \infty. \quad (3.7)$$

With the definition of $k$ in (1.3), we let $\alpha = 1/2k$ in (2.29), and let

$$\tilde{A}(n, \alpha) = [x_0-\delta, x_0+\delta] - A(n, \alpha). \quad (3.8)$$

Then, for all $x \in \tilde{A}(n, \alpha)$

$$\sup\{|n^{1/2}(x[1-F_n(x)] - x[1-F(x)])|_{x \in \tilde{A}(n, \alpha)}\}$$

$$= \sup\{(x_0-x)n^{1/2}[F_n(x)-F(x)] + x \in [x_0, x_0] : x \in \tilde{A}(n, \alpha)\} + o(n^{-1/2})$$

$$= O(n^{-\alpha/2}[\log n]), \text{ with probability 1, as } n \rightarrow \infty, \quad (3.9)$$

by virtue of theorem 2.3, (2.27) and (2.29). Consequently,

$$n^{1/2}[\sup\{x[1-F_n(x)] : x \in \tilde{A}(n, \alpha)\} - \theta] + M_n = O(n^{-\alpha/2}(\log n)), \quad (3.10)$$

with probability 1, as $n \rightarrow \infty$. Finally, by (2.28), as $n \rightarrow \infty$,
it follows from (3.11) and (3.12) that with probability 1, and by (1.3) and the value of \( \alpha = 1/(2k) \),

\[
(3.11) \quad \text{Sup}\{ |x[F^*(x)-F(x)]|: x \in \mathbb{A}(n,\alpha) \} = \mathcal{O}(n^{-\frac{1}{2}}/\sqrt{\log n}),
\]

with probability 1, and by (1.3) and the value of \( \alpha = 1/(2k) \),

\[
(3.12) \quad \text{Sup}\{x[1-F(x)]: x \in \mathbb{A}(n,\alpha) \} \leq \theta - \mathcal{O}([n^{-\frac{1}{2}}(\log n)^k]).
\]

Since for \( n \) sufficiently large, \( O(n^{-\frac{1}{2}}/\sqrt{\log n}) = o(1)O(n^{-\frac{1}{2}}(\log n)^k) \) for all \( k \geq 1 \), it follows from (3.11) and (3.12) that

\[
(3.13) \quad \text{Sup}\{x[1-F^*(x)]: x \in \mathbb{A}(n,\alpha) \} \leq \theta - \mathcal{O}([n^{-\frac{1}{2}}(\log n)^k]),
\]

with probability 1, as \( n \to \infty \). Since, by (2.27), \( M_n = O(\sqrt{\log n}) \), with probability 1 (as \( n \to \infty \)), it follows from (3.7), (3.10) and (3.13) that as \( n \to \infty \),

\[
(3.14) \quad W_n = n^{\frac{1}{2}}(Z_n - \theta) + M_n \to 0, \text{ with probability 1. Q.E.D.}
\]

Now, from (3.14), it follows that for every \( \varepsilon > 0 \), there exists a \( \eta > 0 \), such that for all \( n \geq n_0(\varepsilon, \eta) \)

\[
(3.15) \quad P\{|W_{n+j}| > \varepsilon \text{ for at least one } j=1, \ldots, N, \ldots\} < \eta.
\]

Hence, from theorem 2.4, theorem 3.2 and (3.15), we arrive at the following.

Theorem 3.3. For every positive \( \varepsilon \) and \( \eta \) there exists a \( \delta > 0 \), such that for all \( n \geq n_0(\varepsilon, \eta) \) and \( N < \delta n \)

\[
P\{|Z_{n+j} - Z_n| < n^{-\frac{1}{2}}\xi \varepsilon \text{ for all } j=1, \ldots, N\} \geq 1-\eta.
\]

Further, we state the following result already proved in [3, 5]: under the sole assumption (1.1), for all \( t: -\infty < t < \infty \),

\[
(3.16) \quad \lim_{n \to \infty} P\{ n^{\frac{1}{2}}(Z_n - \theta)/\xi \leq t \} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp(-\frac{1}{2}x^2) dx, \quad -\infty < x < \infty.
\]

Theorem 3.4. If \( \lambda < \infty \), for every positive \( \varepsilon \) and \( s \), there exists a positive \( C_s \) and an \( n_0(\varepsilon, s) \), such that for \( n \geq n_0(\varepsilon, s) \),
Hence, (3.17) follows from (3.19), (3.20), (3.21) and theorems 2.1 and 3.1, while

\[
P(\rho_n > \pi_0 + \epsilon) \leq C_\delta n^{-s}.
\]

If \( E(e^{tx}) \leq E(e^{\epsilon x}) \) for some \( t > 0 \), then for \( n \) sufficiently large

\[
P(\rho_n > \pi_0 + \epsilon) \leq 2(\rho(\epsilon))n, \quad \text{where } \rho(\epsilon) < 1.
\]

**Proof.** Corresponding to every \( \epsilon > 0 \), we can find a \( \eta(\epsilon) > 0 \) such that \( F(x + \eta) = \pi_0 + \epsilon / 2 \).

Then, we have

\[
P(\rho_n > \pi_0 + \epsilon) = P\left(\frac{n}{n+1} F(X_n, r_n) > \pi_0 + \epsilon\right)
\]

\[
\leq P\left(F(X_n, r_n) > \pi_0 + \epsilon\right)
\]

(3.18)

\[
P(\rho_n > \pi_0 + \epsilon) \leq 2(\rho(\epsilon))n, \quad \text{where } \rho(\epsilon) < 1.
\]

Now, by theorem 1 of Hoeffding (1963), we have upon noting that \( F(x + \eta) = \pi_0 + \epsilon / 2 \),

(3.19)

\[
P(F(X_n, r_n) > \pi_0 + \epsilon) \leq \exp(-2\pi n \epsilon^2) = [\rho_1(\epsilon)]^n,
\]

where \( \rho_1(\epsilon) < 1 \). Also, \( [X_n, r_n > x + \eta] \Rightarrow X_n, r_n \ [1-F(X_n, r_n)] < \theta - \delta \), where \( \delta > 0 \). Therefore

(3.20)

\[
P(F(X_n, r_n) > \pi_0 + \epsilon) \leq \exp(-2\pi n \epsilon^2) = [\rho_1(\epsilon)]^n,
\]

where \( \rho_1(\epsilon) < 1 \). Also, \( [X_n, r_n > x + \eta] \Rightarrow X_n, r_n \ [1-F(X_n, r_n)] < \theta - \delta \), where \( \delta > 0 \). Therefore

(3.21)

\[
P(F(X_n, r_n) > \pi_0 + \epsilon) \leq \exp(-2\pi n \epsilon^2) = [\rho_1(\epsilon)]^n,
\]

Hence, (3.17) follows from (3.19), (3.20), (3.21) and theorems 2.1 and 3.1, while

(3.18) follows from (3.19), (3.20), (3.21) and theorems 2.2 and 3.1. Q.E.D.
Though we do not need, it can be shown similarly that bounds similar to (3.17) and (3.18) hold for the lower tail \( P\{p_n < \pi_0 - \varepsilon\} \). It is therefore to be noted that if in (3.17), we let \( s' > 1 \) and apply the Borel-Cantelli lemma, we get that

\[
(3.22) \quad p_n = \pi_0 \quad \text{a.s.}
\]

4. Proofs of theorems 1.1 and 1.2. (i) Theorem 1.1. Since, for \( s' > 0 \),

\[
E\{[N(d)]^{s'}\} < \infty \Rightarrow P\{N(d) < \infty\} = 1,
\]

we shall only prove the former result. Again, we do this only when \( s' \) is a positive integer, as the Liapounoff theorem then yields the result for other values of \( s' \). Define

\[
(4.1) \quad Q_n(c) = P\{N(d) > n\}, \quad n=0,1,\ldots; \quad Q_0(d) = 1.
\]

Then, upon noting that \((n+1)^{s'} - n^{s'} \leq (2^{s'} - 1)n^{s'-1}\), we have

\[
(4.2) \quad E\{[N(d)]^{s'}\} \leq 1 + (2^{s'} - 1) \sum_{n=1}^{\infty} n^{s'-1} Q_n(d).
\]

So, it suffices to prove that for \( n \) sufficiently large,

\[
(4.3) \quad n^{s'+\alpha} Q_n(d) \to 0, \quad \text{for some } \alpha > 0.
\]

Now, by (1.7) and (1.8)

\[
(4.4) \quad Q_n(d) = P\{N(d) > n\} = P\{v_n(d) > r, \quad r=1,\ldots,n\}
\]

\[
\leq P\{v_n(d) > n\} = P\{Z_n^2 > (nd^2/a_n^2)(1/p_n - 1)\}
\]

\[
\leq P\{Z_n^2 > (nd^2/a_n^2)(p_n^{-1} - 1), \quad p_n \leq \pi_0 + \beta\} + P\{p_n > \pi_0 + \beta\},
\]

where \( \beta > 0 \). Since \( p_n < \pi_0 + \beta \Rightarrow p_n^{-1} - 1 \geq \beta > p_n^{-1} - 1 \), we have

\[
(4.5) \quad Q_n \leq P\{Z_n^2 > (ng^*)^2 d/a_n^2\} + P\{p_n > \pi + \beta\}.
\]
Now, $a_n \to a(>0)$, as $n \to \infty$. Hence, for every $d$, there exists an $n_o(d)$ such that for $n \geq n_o(d)$, $(n\delta)^{k}d/a_n > \theta + \varepsilon$, where $\varepsilon > 0$. Hence, from (4.5), we have for all $n \geq n_o(d)$,

$$Q_n \leq P[Z_n > \theta + \varepsilon] + P[p_n > \pi + \beta].$$

Thus, (4.3) directly follows from (4.6), theorem 3.1 and theorem 3.4. Similarly, for $t' \in (-\infty, t_o)$ where $t_o > 0$,

$$E(e^{t'N(d)}) \leq 1 + |e^{t'} - 1| \sum_{n=1}^{\infty} e^{nt'}Q_n(d),$$

and hence, we are only to show that if $E(e^{tX}) < \infty$ for some $t > 0$,

$$Q_n(d) \leq \rho^n$$

for some $\rho < 1$ and all $n \geq n_o$,

where we select $t_o$ in such a way that $\rho e^{t_o} < 1$. (4.8) again follows from (4.6), theorem 3.1 and theorem 3.4. Q.E.D.

(i) Theorem 1.2. From (1.7), (3.5) and (3.22), it follows that

$$y_n = v_n(d)/v(d) = 1 \text{ a.s., uniformly in } d.$$

Thus, upon writing $f(n) = na_n^2/a_n^2$ and $t = a_n^2v(d)$, it follows from (4.9) that the conditions of lemma 1 of Chow and Robbins (1965) are all satisfied. Also, by virtue of (3.16) and theorem 3.3, the conditions (C1) and (C2) of Anscombe (1952) are also satisfied for $\{Z_n\}$, and these conditions are implicit in [4]. Hence, the proof of (1.9) follows along the lines of lemma 1 [and proofs of their (4) and (5)] of Chow and Robbins (1965).

To prove (1.10), we require to show as in lemma 2 of [4] that $E(\text{Sup}_n y_n) < \infty$ or verify the conditions of their lemma 3. Now, by definition,

$$y_n = (a_n^2/a^2)([1-\pi]/\pi \theta^2)[Z_n p_n/(1-p_n)],$$
and hence, it suffices to show that \(E(\sup_n Z_n^2 p_n/(1-p_n)) < \infty\). But,

\[
U_n = Z_n^2 p_n/(1-p_n) = n^{-2}(n-r+1)r X_n^2
\]

\[
\leq (n^{-1}r_n)^{(n^{-1}i=1 r_n X_i^2)} \leq n^{-1}r_n X_1^2 \text{ a.e.,}
\]

and hence, it suffices to show that \(E(\sup_n n^{-1}i=1 X_i^2) < \infty\). A sufficient condition for this is, of course, that \(E(X^4) < \infty\). However, as in lemma 3 of [4], we prove (1.10) without unnecessarily assuming that \(E(X^4) < \infty\). For this, consider first the following lemma.

**Lemma 4.1.** \(U_n/U_{n-1} \geq (1-n^{-1})^2\) for all \(n \geq 2\).

**Proof.** Given the sample of size \(n-1\) i.e., \(X_{n-1} < \ldots < X_{n-1}, n-1\), \(X_n\) can belong to one of the \(n\) intervals: \(X_{n-1,i-1} < X < X_{n-1,i, i=1, \ldots, n}\), where \(X_{n-1,0} = 0\) and \(X_{n-1,n-1} = \infty\). If \(X_{n-1,i-1} < X < X_{n-1,i}\), we have the following two arrays corresponding to the samples of sizes \(n-1\) and \(n\) respectively:

\[
(n-1)X_{n-1,1}, \ldots, (n-1)X_{n-1,i-1}, (n-1)X_{n-1,i}, \ldots, X_{n-1,n-1};
\]

\[
nX_{n-1,1}, \ldots, (n+1)X_{n-1,i-1}, (n+1)X_{n-1,i}, (n+1)X_{n-1,i}, \ldots, X_{n-1,n-1}, X_{n-1,n};
\]

where \((n-r_{n-1}) X_{n-1,r}, r_{n-1}\) and \((n-r+1) X_{n-1,r}, r_{n-1}\) are the maximum values within the first and the second rows. Thus, (i) if \(r_{n-1} \geq 1\), we may have either \(r_n = r_{n-1} + 1 \geq 1\), or \(r_n = n-1\). In the first case, \(X_{n-1,n}, r_{n-1} \geq X_{n-1,r_{n-1}}\) and hence, \(r_n (n-r+1) X_{n-1,n}, r_{n-1}\).

\[
r_n (n-r+1) X_{n-1,n}, r_{n-1} = (n-r_{n-1}) X_{n-1,n};
\]

Thus, \(r_n (n-r+1) X_{n-1,n}, r_{n-1} \geq r_{n-1} (n-r_{n-1}) X_{n-1,n}, r_{n-1}\). Hence, in either case,

\[
(4.12)\quad r_n (n-r+1) X_{n-1,n}, r_{n-1} \geq r_{n-1} (n-r_{n-1}) X_{n-1,n}, r_{n-1};
\]

(ii) If \(r_{n-1} \leq i-1\), it is quite evident that \(r_n = r_{n-1} \geq X_{n-1}, r_{n-1} \geq X_{n-1}, r_{n-1}\) and \((n-r+1) X_{n-1,n}, r_{n-1} \geq (n-r_{n-1}) X_{n-1,n}, r_{n-1}\). Consequently, (4.12) holds again. Thus,
\[ \frac{U_n}{U_{n-1}} = \frac{(n-1)^2}{n^2} \left[ \frac{r_n(n-r+1)X_n^2}{r_{n-1}(n-r-1)X_{n-1}^2} \right]^2 \geq (1 - \frac{1}{n})^2. \text{ a.e.} \]

Hence the lemma.

Using now Lemma 4.1, (4.10) and the inequality (4.11), the proof of (1.10) follows exactly on the same line as in [4, pp. 630-631]. Hence the proof is completed.

5. Fixed percentage error confidence bands for \( \theta \). For every \( d > 0 \), we define 
\[ d_1 = \exp(-d) \text{ and } d_2 = \exp(d), \text{ so that } 0 < d_1 < 1 < d_2 < \infty. \]
We intend to determine \( n \) in such a way that

\[ P\{d_1 Z_n < \theta < d_2 Z_n | \theta \} \sim \gamma(0 < \gamma < 1). \] (5.1)

For this, we consider the following sequential procedure: continue sampling until the first \( p_n < nd^2\{nd^2 + a_n^2\}^{-1} \), (where \( p_n \) and \( a_n \) are defined by (1.5) and (1.6)), and then compute \( Z_n \). By analogy with (1.8), we define the stopping variable \( N(d) \) by

\[ N(d) = \text{Smallest integer } n(\geq 1) \text{ for which } p_n < nd^2\{nd^2 + a_n^2\}^{-1}. \] (5.2)

We also define \( \pi_0 \) as in (1.2) and let

\[ \nu(d) = a_n^2 \pi_0 \{1 - \pi_0\} d^2 \}^{-1}. \] (5.3)

**Theorem 5.1.** The results of Theorems 1.1 and 1.2 also hold for the sequential procedure in (5.2).

**Proof.** Note that

\[ p_n \leq nd^2\{nd^2 + a_n^2\}^{-1} \iff p_n(1 - p_n)^{-1} \leq nd^2 a_n^{-2}. \] (5.4)
Hence, if we define $Q_n(d)$ as in (4.1), to prove the results parallel to those in theorem 1.1, we are only to show that (4.3) and (4.8) hold for the stopping variable $N(d)$, defined by (5.2). Now, by (5.2) and (5.4),

\[
Q_n(d) = P\{N(d)>n\} = P\{p_r(1-p_r)^{-1} > rd^2a_r^{-2}, r=1,\ldots,n\}
\]

\[
= P\{p_n(1-p_n)^{-1} > nd^2a_n^{-2}\} = p_n > nd^2(nd^2+a^2)^{-1}\}.
\]

Now, for every $d>0$, there exists a $n_o$, such that $nd^2(nd^2+a^2) > \pi_o+\epsilon$, for all $n>n_o$, where $\epsilon>0$. Hence, for $n>n_o$,

\[
Q_n(d) \leq P\{p_n > \pi_o+\epsilon\},
\]

and hence, (4.3) or (4.8) follows directly from theorem 3.4.

To prove the results parallel to those in theorem 1.2, we use (3.16) and some standard results on transformation of statistics, and obtain that

\[
\delta_n^{1/2}\log [Z_n/\theta] \rightarrow N(0, \pi_o(1-\pi_o)^{-1}).
\]

Also, by (1.1), $\theta$ is strictly positive, and hence, from (3.5) and theorem 3.3 it follows that for every positive $\epsilon$ and $\eta$, there exists a $\delta>0$, such that for all $n>n_o(\epsilon,\eta)$ and $N<\delta n$,

\[
P{n^{1/2}\log Z_n+j - \log Z_n < \epsilon\{\pi_o/(1-\pi_o)\}^{1/2}, j=1,\ldots,N} > 1-\eta.
\]

Hence, $\{\log Z_n\}$ satisfies both the conditions (C1) and (C2) of Anscombe (1952).

We also note that by virtue of (3.22)

\[
y_n = p_n(1-\pi_o)/(1-p_n)\pi_o = 1 \text{ a.s.},
\]

and hence, using (5.2), (5.4), (5.7)-(5.9) and lemma 1 of [4], it follows that (1.9) holds for the stopping rule (5.2). To prove (1.10), we proceed as follows.
To prove (1.10), we proceed as follows. Let, for some arbitrarily small \( \varepsilon > 0 \),

\[
(5.10) \quad n_i(d) = [\nu(d)(1+(-1)^i\varepsilon)+1], \quad i=1,2,
\]

where \( \nu(d) \) is defined by (5.3) and \([s]\) denotes the integral part of \( s \). Then,

\[
(5.11) \quad E[N(d)/\nu(d)] = [\nu(d)]^{-1}\sum_{n=1}^{\infty} n P[N(d) = n],
\]

where the summation \( \Sigma_1 \) extends over \( n < n_1(d) \), \( \Sigma_2 \) over \( n_1(d) < n < n_2(d) \) and \( \Sigma_3 \) over \( n > n_2(d) \). Since \( \lim_{d \to 0} \nu(d) = \infty \) and \( N(d)/\nu(d) = 1 \) a.s., for every \( \varepsilon > 0 \), we can find a value \( d_0(>0) \), such that for \( 0 < d < d_0 \), \( P[n_1(d) < N(d) < n_2(d)] = P[1-\varepsilon < N(d)/\nu(d) < 1+\varepsilon] = 1-\eta_1 \), where \( \eta_1(>0) \) is a preassigned small quantity. Then, for \( d < d_0 \), the first sum on the r.h.s. of (5.11) is less than

\[
(5.12) \quad (1-\varepsilon)P[N(d) \leq n_1(d)] \leq \eta_1 (1-\varepsilon) \leq \eta_1
\]

The second sum is equal to \( 1+R(d) \), where

\[
(5.13) \quad |R(d)| \leq \varepsilon P[n_1(d) < N(d) < n_2(d)] + [1-P[n_1(d) < N(d) < n_2(d)]]
\]

\[
\leq \varepsilon + \eta_1.
\]

Finally, on using (5.5), we have

\[
(5.14) \quad [\nu(d)]^{-1}\sum_{n_2(d)}^{\infty} n P[N(d) = n] = [\nu(d)]^{-1}\sum_{n_2(d)}^{\infty} n P_N(d)
\]

\[
\leq [\nu(d)]^{-1}\sum_{n_2(d)}^{\infty} P[p_n > nd^2(nd^2+a^2)^{-1}]
\]

\[
\leq [\nu(d)]^{-1}\sum_{n_2(d)}^{\infty} P[p_n > \pi_o(1+\varepsilon)(1+\varepsilon)(1-\pi_o)a_n^2/a^2]^{-1}
\]

\[
\leq [\nu(d)]^{-1}\sum_{n_2(d)}^{\infty} P[p_n > \pi_o + \varepsilon'], \text{ where } \varepsilon' > 0.
\]

Hence, if we use theorem 3.4 (where in (3.17), we let \( s > 1 \)), it follows that the r.h.s. of (5.14) can be bounded above by \( C_1 + \gamma [\nu(d)]^{-1}[n_2(d)]^{-\gamma} \), \( \gamma > 0 \), which by (5.10), converges to zero as \( d \to 0 \). Hence, the result follows from (5.11) through (5.14). Q.E.D.
REFERENCES


