ROBUSTNESS OF CERTAIN TESTS OF CENSORING OF EXTREME SAMPLE VALUES
II: SOME EXACT RESULTS FOR EXPONENTIAL POPULATIONS* 

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Institute of Statistics Mimeo Series No. 940
July 1974
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1. Introduction

In [1] we have given general formulae for evaluating the effect of an incorrect choice of population distribution on certain tests of sample censoring. (For definitions, etc. see [2][3].)

These formulae apply when the form of population distribution is known, but certain parameters $\theta$ have to be estimated. The formulae apply when possibly incorrect values $\theta^*$ are used for $\theta$.

When $\theta$ has to be estimated, there will usually be random variation in $\theta^*$. The overall effect of this can be evaluated by taking expected values of the formulae in [1] with respect to suitable distributions for $\theta^*$.

We shall suppose throughout the paper that there is sampling from a population in which the measured character has an absolutely continuous distribution.

2. General Discussion

As in [1] and earlier papers we suppose $r$ values $X_1 \leq X_2 \leq \ldots \leq X_r$ are available and wish to test whether they are a complete random sample, or come from a complete random sample of size $(s_0 + r + s_r)$, from which the $s_0$ smallest and $s_r$ largest values have been censored. We denote this last hypothesis by $H_{s_0, s_r}$; $H_{0,0}$ corresponds to the case of no censoring.
The critical regions of the tests described in [1][2] are all defined in terms of the probability integral transforms

\[ Y_1 = \int_{-\infty}^{X_1} f(x|\tilde{\theta}) \, dx \quad \text{and} \quad Y_\tau = \int_{-\infty}^{X_\tau} f(x|\tilde{\theta}) \, dx \]

of the minimum and maximum values \( X_1, X_\tau \) respectively. Here \( f(x|\tilde{\theta}) \) is the density function of \( X \). If, in fact, an incorrect value of \( \tilde{\theta} \) is used - \( \tilde{\theta}^*, \) say - then instead of calculating \( Y_1, Y_\tau \) we actually calculate

\[ Y_j^* = \int_{-\infty}^{X_j} f(x|\tilde{\theta}^*) \, dx \quad (j = 1, \tau) . \]

Now provided \( f(x|\tilde{\theta}) > 0 \) for all \( x \) and \( \tilde{\theta} \), (1) and (2) define \( Y_j^* \) as a unique function \( h^*(Y_j) \) of \( Y_j \). So when we use a critical region \( w(k) \) defined by

\[ w(k) \{ Y_1^*, Y_\tau^* \} \geq C^{(k)}_\alpha \]

( \( C^{(k)}_\alpha \) is a constant which would give 100\(\alpha\)% significance level if \( \mathcal{H}_0,0 \) were valid and the correct \( \tilde{\theta} \) were used) we are actually using a critical region \( w^*(k) \) defined by

\[ w^*(k) \{ h^*(Y_1), h^*(Y_\tau) \} \geq C^{(k)}_\alpha \]

in terms of the correct \( Y_1 \) and \( Y_\tau \). The latter have joint density function (under \( \mathcal{H}_{0,0} \)).
For a given $\theta^*$, the power $\beta_k(s_0, s_r | \theta^*)$ of the test with critical region $w^*_k$ is the integral of (4) over the region

$$w^*_k \{ h^* (y_1), h^* (y_r) \} \geq c^{(k)}_\alpha.$$  

The unconditional, overall power $\beta_k(s_0, s_r)$ is the expected value, with respect to variation in $\theta^*$ of $\beta_k(s_0, s_r | \theta^*)$. In particular the actual significance level is

$$\beta_k(0, 0) = r(r - 1)E \left[ \int_{w^*_k} \int (y_r - y_1)^{r-2} dy_1 dy_r \right].$$  

($\theta^*$, of course appears implicitly in $w^*_k$.) This will not in general be equal to $\alpha$, if $c^{(k)}_\alpha$ is calculated in accordance with the formulae in [2]. It will, in particular cases, be possible to choose $c^{(k)}_\alpha$ so that the overall significance level is $\alpha$, but the necessary formulae will depend on the density function of $y_r$, and on that of $\theta^*$, while the values in ( ) do not.

As a concrete example we consider the case of normal variation.

In [1] it was shown that if $\theta = (\xi, \sigma)$ and

$$f(x | \xi, \sigma) = (\sigma \sqrt{2\pi})^{-1} \exp \left( - (x - \xi)^2 (2\sigma^2)^{-1} \right)$$

then $\beta(s_0, s_r | \xi^*, \sigma^*)$ depends only on
\[ \Delta^* = (\xi^* - \xi)/\sigma \quad \text{and} \quad \epsilon^* = \sigma^*/\sigma. \]

If \( \xi^* \) and \( \sigma^2 \) are calculated from an independent random sample of size \( N \), as the sample mean and variance respectively then:

(i) \( \Delta^* \) and \( \epsilon^* \) are independent

(ii) \( \Delta^* \) is distributed \( \mathcal{N}(\xi, N^{-1}) \) which gives useful practical indications

(iii) \( \epsilon^* \) is distributed \( (N - 1)^{-1/2} \chi_{N-1} \).

This information is sufficient to enable us to calculate

\[
\beta(s_0, s_r) = E[\beta(s_0, s_r|\xi^*, \sigma^*)]
\]

the expectation being with respect to variation of \( \xi^* \) and \( \sigma^* \).

Although this may be evaluated explicitly using numerical cubature applied to \( \beta(s_0, s_r|\xi^*, \sigma^*) \) which is itself quite a complicated function, the calculations are quite heavy even with the assistance of an electronic computer. (It is hoped to provide some numerical tables in the third report in this series.) It is rather easier to use a two-entry table of values of \( \beta(s_0, s_r|\xi^*, \sigma^*) \) and (very roughly) average with respect to the distributions of \( \xi^* \) and \( \sigma^* \).

A further possible complication is that it may be necessary to estimate \( \theta \) from the \( r \) values \( X_1, \ldots, X_r \) actually available, so that \( \theta^* \) and \( (Y_1, Y_r) \) are no longer mutually independent. We do not consider this situation in the present paper, except for a brief discussion in the final
section.

3. Exponential Populations

The main aim of the present paper is to evaluate performance of the three tests described in [2] when applied to a population in which the (unorthodox) $X$'s have an exponential distribution with expected value $\theta$. We will suppose $\theta$ to be estimated as the arithmetic mean of a random sample (independent of the $r$ values to be analysis) of size $n$, so that:

$$\theta^* / \theta \quad \text{is distributed as} \quad (2n)^{-1} \chi^2_{2n}. $$

Under these conditions it is possible to get explicit formulae for properties of the tests which are relatively simple (as compared, for example, with the normal case).

This discussion will, it is hoped, help to make clearer the general description in Section 2.

The population density function is $\theta^{-1} \exp(-\theta/x) \ (\theta, x > 0)$ corresponding to a $\frac{1}{2} \theta^2 \chi^2_2$ distribution.

We have

$$Y = \int_0^\infty \theta^{-1} \exp(-\theta x) dx = 1 - e^{-x/\theta} \ (x > 0).$$

Hence

$$Y^* = 1 - e^{-x/\theta^*} = 1 - (1 - Y)^{\theta/\theta^*}. $$

The critical regions to be used in testing for extreme censoring (1) form below (2) symmetrically or (3) in general, are (see [2]) respectively.
(10.1) (Below) \[ Y_1 \geq C^{(1)}_\alpha \]

(10.2) (Symmetric) \[ Y_1(1 - Y_r) \geq C^{(2)}_\alpha \]

(10.3) (General) \[ Y_1 + (1 - Y_r) \geq C^{(3)}_\alpha \]

If an incorrect value, \( \theta^* \), is used for \( \theta \), the regions actually used are, respectively

(11.1) \[ 1 - (1 - Y_1)^{\theta/\theta^*} \geq C^{(1)}_\alpha \]

(11.2) \[ \{1 - (1 - Y_1)^{\theta/\theta^*}\} (1 - Y_r)^{\theta/\theta^*} \geq C^{(2)}_\alpha \]

(11.3) \[ 1 - (1 - Y_1)^{\theta/\theta^*} + (1 - Y_r)^{\theta/\theta^*} \geq C^{(3)}_\alpha \]

4. Censoring from Below

The inequality (11.1) can also be written

(11.1)': \[ Y_1 \geq 1 - (1 - C^{(1)}_\alpha)^{\theta^*/\theta} = 1 - (1 - C^{(1)}_\alpha)^T \]

where we put

(12) \[ T = \theta^*/\theta \]

Since; given \( H_{s_0, s_T} \), the distribution of \( Y_1 \) is standard beta with parameters \( s_0 + 1, r + s_T \) we have
\[ (13) \quad \beta(1) (s_0, s_r | \theta^*) = \Pr[Y_1 \geq 1 - (1 - c^{(1)}_{\alpha}) T | s_0, s_r] \]
\[ = I (1-c^{(1)}_{\alpha}) T (s_r + s_0 + 1) \]

where \( I_p(\alpha, \gamma) = \{B(\alpha, \gamma)\}^{-1} \int_0^p t^{\alpha-1} (1 - t)^{\gamma-1} \, dt \) is the incomplete beta function ratio.

In particular the conditional significance level is

\[ (14) \quad \beta(1)(0, 0 | \theta^*) = I (1-c^{(1)}_{\alpha}) T (r, 1) = (1 - c^{(1)}_{\alpha}) r T . \]

The unconditional significance level is

\[ (15) \quad E[\beta(1)(0, 0 | \theta^*)] = E[(1 - c^{(1)}_{\alpha}) T] \]
\[ = E \left[ (1 - c^{(1)}_{\alpha}) \chi^2_{N} \right] \]
\[ = \{1 - r^{N-1} \log(1 - c^{(1)}_{\alpha})\}^{-N} . \]

In order to make the significance level equal to \( \alpha \), we must choose \( c^{(1)}_{\alpha} (= c^{(1)}_{\alpha}(N), \text{say}) \) to satisfy

\[ \{1 - r^{N-1} \log(1 - c^{(1)}_{\alpha})\}^{-N} = \alpha \]

i.e.

\[ (16) \quad c^{(1)}_{\alpha}(N) = 1 - \exp\left[ -1/(1 - \alpha^{-1}N) \right] . \]
If the value \( C^{(1)}_\alpha = 1 - \alpha^{1/r} \), which is appropriate when \( \theta \) is known, is used then the overall significance level is actually

\[
(17) \quad (1 - N^{-1} \log \alpha)^{-N}.
\]

As would be expected this tends to \( \alpha \) as \( N \) tends to infinity.

The power with respect to \( H_{S_0, 0} \) (i.e. actual censoring from below) is

\[
(18) \quad \beta_{(1)}(s_0, 0) = E \left[ \frac{r+s_0}{(r-1)! s_0!} \sum_{j=0}^{s_0} (-1)^j {s_0 \choose j} (r+j)^{-1} \left\{ (1-C^{(1)}_\alpha)^{(r+j)N} - 1 \right\} \right].
\]

Inserting the value for \( C^{(1)}_\alpha \) from (16) (which gives overall significance level equal to \( \alpha \)) we obtain

\[
(19) \quad \beta_{(1)}(s_0, 0) = \frac{(r+s_0)!}{(r-1)! s_0!} \sum_{j=0}^{s_0} (-1)^j {s_0 \choose j} (r+j)^{-1} \{ 1 + (1+jr^{-1})(\alpha^{-1/N}-1) \}^{-N}.
\]

Table 1 gives values of \( \beta_{(1)}(s_0, 0) \) for \( r = 5(5)25, N = 5, 10, 25, 50 \) and \( s_0 = 2(2)8 \) with \( \alpha = 0.05 \). These are powers when the actual significance level is 0.05. Table 2 gives the actual significance levels and powers obtained if the value \( C^{(1)}_\alpha = 1 - \alpha^{1/r} \) is used. In this case
5. Symmetrical and General Censoring

Similar calculations for the tests with critical regions (11.2) and (11.3) are complicated by the fact that both $Y_1$ and $Y_r$ appear in the inequalities. In these cases we use the fact that the joint density functions of $Z_1 = 1 - Y_1$ and $Z_r = 1 - Y_r$ under $H_{0, r}$ is

$$
\frac{(r+s_0+s_r)!(r-1)!s_0!}{s_0!(r-2)!s_r!} (1 - z_1)^{s_0} (z_1 - z_r)^{r-2} z_r^{s_r} \quad (0 \leq z_r \leq z_1 \leq 1)
$$

and so the conditional power $\beta_{(k)}(s_0, s_r; \theta^*)$ is obtained by integrating (21) over the regions

$$
(22.2) \quad 1 - z_1^{\theta/\theta^*} z_r^{\theta/\theta^*} \geq C^{(2)}_\alpha \quad \text{for } k = 2 \quad \text{(symmetrical censoring)}
$$
or

$$
(22.3) \quad 1 - z_1^{\theta/\theta^*} + z_r^{\theta/\theta^*} \geq C^{(3)}_\alpha \quad \text{for } k = 3 \quad \text{(general censoring)}
$$

The resultant value has in each case, to be averaged over the distribution $\theta^*$ (or equivalently of $T = \theta^*/\theta$).

Making the transformations $v_j = z_j^{\theta/\theta^*}$ ($j = 1, 2$) we have

$$
\beta_{(k)}(s_0, s_r; \theta^*) = \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!} \int \int \tau^r (1-v_1^r) v_1^{s_0} (1-v_r^r) v_r^{s_r+1} d\tau dv_1 dv_r
$$

(Some values of $\beta_{(1)}(s_0, s_r)$ with $s_r = 0$ are also given.)
The regions of integration are:

(24.2) For \( k = 2 \): \( (1 - v_1) v_r \geq C(2)_a ; 0 \leq v_r \leq v_1 \leq 1 \)

(24.3) For \( k = 3 \): \( 1 - v_1 + v_r \geq C(3)_a ; 0 \leq v_r \leq v_1 \leq 1 \).

The overall power is obtained by taking the expected value with respect to the distribution of \( T \), which is \( (2^N)^{-1} X_{2N}^2 \).

By performing the integration first with respect to \( T \), and then with respect to \( v_r \), we can reduce the triple integral to a finite sum of single integrals.

The integrand in (23) is

\[
T^2 \left\{ \sum_{i=0}^{s_0} \left( \begin{array}{c} s_0 \\ i \end{array} \right) (-1)^i \frac{1}{v_1} i^T \right\} \left\{ \sum_{j=0}^{r-2} \left( \begin{array}{c} r-2 \\ j \end{array} \right) (v_1 v_r)^j T^{r-2-j} \frac{1}{v_2} \right\} \frac{v_r (s_r+1) T}{v_1 v_r} (v_1 v_r)^{-1}
\]

\[
= (v_1 v_r)^{-1} T^2 \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^i j \frac{1}{v_1} i^T \left( \begin{array}{c} r-2 \\ j \end{array} \right) v_1 (s_r+j+1) T
\]

The expected value of this with respect to variation in \( T \) is

\[
(v_1 v_r)^{-1} \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^i j \frac{1}{v_1} i^T \left( \begin{array}{c} r-2 \\ j \end{array} \right) (2^N)^{-1} \times
\]

\[
\times E \left[ X_{2N}^2 \exp \left( (2^N)^{-1} [(r+i-j+1) \log v_1 + (s_r+j+1) \log v_r] X_{2N}^2 \right) \right]
\]

(25)

\[
= (v_1 v_r)^{-1} (N+1) \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^i j \frac{1}{v_1} i^T \left( \begin{array}{c} r-2 \\ j \end{array} \right) \times
\]

\[
\times [1 - (-1)^i (r+i-j-1) \log v_1 + (s_r+j+1) \log v_r]^{-1} \times \]

\[
\times [1 - (-1)^i (r+i-j-1) \log v_1 + (s_r+j+1) \log v_r]^{-1} (N+2) .
\]
Using the relation $E[ X^4 Y \exp(A Y^2)] = v(v + 2)(1 - 2A)^{-\frac{v-2}{2}}$ for $A < \frac{1}{2}$.

The indefinite integral of (25) with respect to $v_r$ is

$$v_l^{-1} \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^{i+j} \binom{s_0}{i} \binom{r-2}{j} (s_r + j + 1)^{-1} \times$$

$$[1 - i^{-1} (r + i - j - 1) \log v_l + (s_r + j + 1) \log v_r]^{-i+m_1}.$$  

For $k = 2$, the range of integration for $v_r$ is $(1 - v_l)^{-1} C^{(2)}_a s v_r \leq v_l$ and for $v_1$, $v_1(1 - v_1) \geq C^{(2)}_a$ i.e. $V_l^1 \leq v_1 \leq V_1^+$ where

$$V_1^\pm = \frac{1}{2} \left[ 1 \pm \sqrt{1 - 4C^{(2)}_a} \right].$$

Hence

(27.2) $\beta^{(2)}(s_0, s_r) = \frac{(r+s_0+s_r)!}{s_0! (r-2)! s_r!} \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^{i+j} \binom{s_0}{i} \binom{r-2}{j} (s_r + j + 1)^{-1} \times$

$$\times [r + s_r + i]^{-1} \left\{ [1 - i^{-1} (r + s_r + i) \log V_1^+]^{-i+m_1} - [1 - i^{-1} (r + s_r + i) \log V_1^-]^{-i+m_1} \right\}$$

$$\int_{V_1^-}^{V_1^+} [1 - (s_r + j + 1) \log (1 - v_1) + (r + i + j - 1) \log v_1]^{i+m_1} \frac{C^{(2)}_a}{1 - v_1} dv_1.$$  

When $k = 3$, the range of integration for $v_r$ is $v_l - (1 - C^{(3)}_a) s v_r \leq v_l$, and the range of integration for $v_1$ is $1 - C^{(3)}_a \leq v_1 \leq 1$.

Hence
6. Comments on Tables

Note that in all cases, for $s_0 + s_r$ constant, the power decreases as $s_0$ decreases. This is to be expected, since this test is designed to detect alternative hypotheses of the hypothesis. Table I shows that the use of $e(1) = 1 - \alpha$ as critical limit increases the significance level, and also the power. As might be expected, this effect decreases as $N$ increases.

If $C^{(1)}(N)$ is used so that the actual significance level is $\alpha$, Table 2 shows that the power increases with $N$. This is not always the case in Table I. Only for $s_0 + s_r$ sufficiently big, and $s_0$ not too small, does the power increase for $N$ in Table 1.

As $N \to \infty$ the values in both Tables tend to the values for the case when the population distribution is known, (which do not depend on the actual distribution, so long as it is known and continuous).

I would like to express my gratitude to Mr. Kerry L. Lee for programming the computation of extensive tables of which these are a part.
7. No Prior Estimate of Parameter Available

In this case we may (assuming the null hypothesis \( H_{0,0} \) to be valid) estimate \( \theta \) by \( \theta^* \), the arithmetic mean of the \( r \) available \( X \)'s. If \( H_{0,0} \) is valid then \( \theta^* \) is distributed as \((2r)^{-1}\theta^2\), but \( X_1, X_2, \ldots, X_r \) are not independent of \( \theta^* \). In fact

\[
Pr[X_1(r\theta^*)^{-1} < k | H_{0,0}] = 1 - r(1 - K)^{r-1} + \left\{ \begin{array}{l} \frac{r}{2} \\ (1 - 2K)^{r-1} \end{array} \right. 
\]

(the series terminates at the \( m \)-th term when \( 1 - mk < 0 \leq 1 - (n - 1)K \)).

In this case, the actual significance level for \( k = 1 \) (i.e. when testing for censoring from below) is

\[
Pr[Y_1 > C^{(1)}_\alpha | H_{0,0}] = \Pr[X_1(r\theta^*) > -r^{-1} \log(1 - C^{(1)}_\alpha)]
\]

\[
= r(1+r^{-1}\log(1-C^{(1)}_\alpha))^{r-1} - \left\{ \begin{array}{l} \frac{r}{2} \\ \{1+2r^{-1}\log(1-C^{(1)}_\alpha)\}^{r-1} \end{array} \right. 
\]

(28)

It is possible to choose \( C^{(1)}_\alpha \) to make the actual significance level equal to \( \alpha \) but we cannot give an explicit formula, as in (16).

REFERENCES


TABLE 1

Power w.r.t. $H_{s_0,s_r}$ of (11.1) with $C^{(1)}_\alpha = \frac{1}{1 - \alpha^r}$ ($\alpha = 0.05$)

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## TABLE 2

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