EXISTENCE AND EXPLICIT DETERMINATION
OF OPTIMAL STOPPING TIMES

by

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Abstract

We consider large classes of continuous time optimal stopping problems for which we establish the existence and form of the optimal stopping times. These optimal times are then used to find approximate optimal solutions for a class of discrete time problems.
Introduction

This paper focuses on three problems in the theory of optimal stopping:

(1) The determination of a broad class of continuous time optimal stopping problems for which optimal stopping times exist. This problem is treated in Section (I).

(2) The explicit determination of existing optimal stopping times. This problem is treated in Section (II).

(3) The connection between weak convergence and optimal stopping, aspects of which are treated in Section (III).

A more detailed survey of the contents is given directly below.

We remark here that we made no use of the "free-boundary" approach to optimal stopping which one finds, for instance, Van Moerbeke and Chernoff.

Let \( X = \{ X_t, t \geq 0 \} \) be a standard Markov process. Let \( f: [0, \infty) \times \mathbb{R} \to \mathbb{R} \) be continuous. Let \( P(t, x) \) represent the conditional probability \( P(\cdot|X_t = x) \). We define the optimal stopping problem for \( X \) and \( f \) as the problem of determining \( F, T_\infty \) where

\[
F(t, x) = \sup_{T \geq t} \int f(T, X_T) dP(t, x) = \int f(T, X_T) dP(t, x)
\]

where \( T \) runs through stopping times and \( T_\infty \), if it exists, is a stopping time that realizes the supremum. We prove in Section (I) that for a broad class of processes \( X \) and continuous functions \( f \) the optimal time \( T_\infty \) exists and is the hitting time of the closed
Our principal result is Theorem (1.3). This result covers at least the following cases:

(A) $X$ is standard Markov, $f(t,x) = c(t)x$ where $c$ is continuous non-increasing and

\[
\begin{align*}
&P(t,x)\left\{ \lim_{s \to \infty} c(s) |X_s| = 0 \right\} = 1, \text{ all } (t,x) \\
&\text{and} \\
&\int_{s \geq 0} \sup_{s \geq 0} c(t+s) |X_{t+s}| dP(t,x) < \infty, \text{ all } (t,x)
\end{align*}
\]

(B) $X$ is standard Markov, $f(t,x) = x - at$ for $a > 0$, $X$ has continuous paths, and

\[
\begin{align*}
&P(t,x)\left\{ \lim_{s \to \infty} X_s - as = -\infty \right\} = 1, \text{ all } (t,x) \\
&\int_{s \geq 0} \sup_{s \geq 0} (X_{t+s} - a(t+s))^+ dP(t,x) < \infty, \text{ all } (t,x)
\end{align*}
\]

(C) $X$ is a pure jump process with positive bounded jumps; $a > 0$, $b > 0$, $g: [0, \infty) \to \mathbb{R}$ is continuous, and $f(t,x) = g(x) - ax - bt$ where

\[
\begin{align*}
&P(t,x)\left\{ \lim_{s \to \infty} g(X_s) - ax_s - bs = -\infty \right\} = 1, \text{ all } (t,x) \\
&\text{and} \\
&\int_{s \geq 0} \sup_{s \geq 0} (g(X_{t+s}) - ax_{t+s} - b(t+s))^+ dP(t,x) < \infty, \text{ all } (t,x)
\end{align*}
\]
In Section (II) we treat the subclass of (A) where \( c(t) = -\lambda t \), \( \lambda > 0 \). We show by elementary arguments that for a class of processes \( X \) which includes, in particular, martingales and processes with stationary independent increments, there exists a positive \( x_0 \) such that \( T_\infty \) is the hitting time of \([x_0, \infty)\) for the process \( X \), i.e.

\[
\Gamma_\infty = \{F = f\} = \{(t,x) : x \geq x_0 \text{ or } t = \infty\}
\]

If \( X \) is a diffusion the value \( x_0 \) can often be characterized as a solution to

\[
xH(\lambda, x) = 1
\]

where \( H \) is a smooth function related to Laplace transforms of first passage times. There results are treated in Theorem (2.1), (2.2), (2.3). Applications are given covering Brownian Motion, Ornstein-Uhlenbeck processes, and Poisson processes. Our solutions in these three cases agree with those obtained previously by Taylor [ ]; our methods are different.

In Section (III) we consider the relationships between weak convergence and optimal stopping. We have a sequence of processes, \( X_n \), converging weakly a process \( X_\infty \). We define

\[
F_n(t,x) = \sup_{T \geq t} \int f(T, X_n(t))dP(t,x), \quad n \leq \infty.
\]

Suppose we can find \( T_\infty \), the optimal stopping time for the case \( n = \infty \) above. Can we use \( T_\infty \) as an approximation to \( T_n \), the optimal stopping time in those cases above where \( n \) is large? An affirmative
answer is given in Theorems (3.2) and (3.3) under restrictions motivated by Theorem (3.1). The latter theorem may be loosely paraphrased as follows: The optimal hitting $T_\infty$ defined by

$$F(t,x) = \sup_{T \geq t} \int_c(T)X_t dP(t,x) = \int_c(T_\infty)X_t dP(t,x)$$

where $X$ is brownian motion and $c$ is decreasing, is given by

$$T_\infty = \inf t \geq X_t \geq G(t)$$

where $G$ is continuous, positive, non-decreasing. Similar results, subject to slight modifications, would hold for most processes $X$ with stationary independent increments. Our methods show, in particular, that when $c(t) = \frac{1}{(A+Bt)^r}$, $r > \frac{1}{2}$, and when $X$ is brownian motion, we have $G(t) = K\sqrt{\frac{A+Bt}{B}}$, $k$ constant.

Our methods yield the square root form almost immediately; the determination of $k$ requires other tools - see Walker [ ] or Shepp [ ].
I. Some Optimal Stopping Theory

Let \( X = \{X_t, t \geq 0\} \) be a real valued Markov process with the following properties

(1.1) \( X \) is strong Markov, right continuous with left limits and \( P_x \{X_0 = x\} = 1, \) all \( x \in \mathbb{R} \).

(1.2) \( X \) has the Feller property, i.e., for all bounded continuous \( f \), \( T^t f \) is continuous where

\[
(T^t f)(x) = \int f(X_t) dP_x.
\]

(1.3) \( X \) is extended quasi-left continuous, i.e., for all increasing sequences of stopping times, \( T_n, \) if \( T_n \uparrow T_\infty \), then \( X_{T_\infty} \) exists \( P_x \) a.e. and

\[
X_{T_n} \rightarrow X_{T_\infty} \quad \text{a.e.} \quad P_x \quad \text{a.e.}, \text{ all } x.
\]

We remark that (1.1), (1.2) and the version of (1.3) restricted to \( X_{T_n} \rightarrow X_{T_\infty} \) on \( \{T_\infty < \infty\} \) defines a standard process; see Dynkin [1].

Let \( Z = \{(t,X_t), t \geq 0\} \) be the space-time version of \( X \) and let \( f \) be non-negative and continuous on \( [0,\infty) \times \mathbb{R} \). We will call \( f \) a return function. We assume \( f \) satisfies

(1.4) For all \( (t,x) \):

\[
\int \sup_{s \geq 0} f(t+s, X_{t+s}) dP(t,x) < \infty
\]

where, of course, the integral is taken as the canonical version of the conditional expectation, i.e.,

\[
\int \sup_{s \geq 0} f(t+s, X_{t+s}) dP(t,x) = E\left\{\sup_{s \geq 0} f(t+s, X_{t+s}) | X_t = x\right\}.
\]

Remarks

We assume throughout this section that \( f(\infty, X_\infty) \) is well-defined \( P(t,x) \) a.e., i.e., \( f \) is continuous at infinity.
Statement of the Problem

To determine the optimal payoff \( F \) and the optimal stopping time \( T_\infty \) where

\[ F(t,x) = \sup_{T \geq t} \int f(T,X_T)dP = \int f(T_\infty,X_{T_\infty})dP(t,x), \]

(1.6) Remarks

(I) The most interesting processes \( X \) will not satisfy (1.3), a hypothesis central to our development. However, a re-definition of \( X \) and \( f \) will often suffice to transform the problem stated in (1.5) into one where \( X \) satisfies (1.3). For example, if \( X \) is standard Brownian motion, and if \( f(t,x) = e^{-\lambda t}x^+ \), then the problem (1.5) is identical to that problem involving the process \( \tilde{X} \) and the function \( \tilde{f} \) where \( \tilde{X}_t = e^{-\lambda t}x_t \) and \( \tilde{f}(t,x) = x^+ \), for now \( \tilde{X}_t \to 0 \) a.e. as \( t \to \infty \) and so (1.3) obtains. This construction will be elaborated on in the next remark.

(II) Let \( X \) be a Markov process which satisfies (1.1), (1.2) and

where

\[ \begin{align*}
(a) \quad & X_t = X_0 + Y_t + Z_t, \quad Y_0 = Z_0 = 0 \\
(b) \quad & X_0 \text{ is square integrable} \\
(c) \quad & Y = \{Y_t, t \geq 0\} \text{ is a martingale where,} \\
& \quad \text{for all } t \geq 0, s \geq 0 \text{ and for some fixed } K > 0: \\
& \quad \mathbb{E}\{(Y_{t+s} - Y_t)^2 | X_t\} \leq Ks \\
(d) \quad & |Z_t| \leq Kt, \text{ all } t \geq 0.
\end{align*} \]

We list two classes of processes which have these properties:
Class (1)

\[ X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t m(X_s) ds \]

where \( \sigma, m \) are bounded, \( W \) is Brownian Motion, \( X_0 \) is square integrable, independent of \( W \).

Class (2)

\[ X = \{X_t, t \geq 0\} \] is pure jump, range the integers, having intensities \( \{\lambda_x\} \) and transition probabilities \( \{Q_x\} \) where, if \( T_x \) is the time for a jump from state \( x \):

\[
\begin{align*}
& P_x\{T_x > t\} = e^{-\lambda_x t} \\
& P_x\{X_{T_x} = x+y\} = Q_x\{y\} \\
& \sup_x \lambda_x < \infty \\
& \sup_x \int y^2 Q_x(dy) < \infty.
\end{align*}
\]

(1.9)

Our concern in later applications will be with determining \( F, T_\infty \) for processes of Class (1) or Class (2) subject to a damping factor, i.e., we consider

\[
F(t,x) = \sup_{T \geq t} \int C(T)X_t dP(t,x) = \int C(T_\infty)X_{T_\infty} dP(t,x)
\]

(1.10)

where \( C \) decreases in \( t \). In order to use our soon to be developed theory on such problems, where, as explained in the last remark, we'll rename our process as \( \tilde{X} = \{C(t)X_t, t \geq 0\} \) and our return \( \tilde{f}(t,x) = x^+ \), we must determine that (1.5), (1.4) hold. We state a result – see Mucci, [ p. 7] for a proof – which gives us what we need. Similar results occur in Walker [ ].
**Proposition (1.1)**

Let $X$ satisfy (1.7) and let $C: [0,\infty) \to [0,\infty)$ be non-increasing. Then

(A) If $Z_t \equiv 0$, all $t \geq 0$ and $\int_0^\infty C^2(t)dt < \infty$, then

$$P(t,x)\left\{\lim_{s \to \infty} c(s)X_s = 0\right\} = 1 \text{ all } (t,x)$$

and

$$\int_{s \geq 0} \sup_{c(t+s)|X_{t+s}|} dP(t,x) < \infty, \text{ all } (t,x)$$

(B) For general $Z$, (1.11) continues to hold provided

$$tc(t) \to 0 \text{ as } t \to \infty.$$

**Development of the Theory**

Let (1.1) through (1.4) hold. We will determine the general form of $F, T_\infty$ defined in (1.5).

**(1.13) Definitions**

A) $G: [0,\infty) \times R \to [0,\infty)$ is called excessive if $G$ is measurable and if, for all $(t,x)$, all $s \geq 0$:

$$G(t,x) \geq (T^sG)(t,x) = \int G(t+s,X_{t+s})dP(t,x)$$

and

$$G(t,x) = \lim_{s \to 0} (T^sG)(t,x).$$

B) An excessive $G$ is called an excessive majorant of $f$ if $G \geq f$.

C) An excessive majorant $G$ of $f$ is called the least excessive majorant if, for all excessive majorants $H$ of $f$, $H \geq G$. 
Grigelionis-Shirgaev [ ] determine that \( f \) has a least excessive majorant, \( f_\infty \), defined by the recursion
\[
f_0 = f, \quad f_n = \sup_{t \geq 0} T^n f_{n-1}, \quad f_\infty = \lim_{n \to \infty} f_n .
\]
Taylor [ ] shows that the continuity of \( f \) and the Feller property (1.2) made \( f_\infty \) lower semi-continuous.

(1.14) Definitions
\[
\Gamma_\infty = \{(t,x): f(t,x) = f_\infty(t,x)\}
\]
\[
T_\infty = \inf t \geq 0 \quad (t,x) \in \Gamma_\infty .
\]

The lower semi-continuity of \( f_\infty \) makes \( \Gamma_\infty \) a closed set, and the hitting time, \( T_\infty \), is well-defined, since \( f_\infty(\cdot, X_\infty) = f(\cdot, X_\infty) \), for we have
\[
\begin{align*}
\Gamma_\infty & = \{(t,x): f(t,x) = f_\infty(t,x)\} \\
& \text{exists } P_{(t,x)} \text{ a.e.}
\end{align*}
\]
and
\[
\begin{align*}
\{f_\infty(\infty, X_\infty) = f(\infty, X_\infty)\} & = 1 .
\end{align*}
\]
A proof of (1.15) can be adapted from Neveu [ ]. We begin with the observation that
\[
\begin{align*}
f_\infty(t,x) & \leq \int_{s \geq 0} \sup s \to s \int f(t,s,X_{t+s})dP(t,x)
\end{align*}
\]
This we can show inductively. Clearly
\[
\begin{align*}
f_0(t,x) & \leq \int_{s \geq 0} \sup s \to s \int f(t+s,X_{t+s})dP(t,x)
\end{align*}
\]
Assume this inequality obtains with \( f_n \) on the left. Then
\[ f_{n+1}(t,x) = \sup_{s \geq 0} \int f_n(t+s,X_{t+s})dP(t,x) \]
\[ \leq \sup_{s \geq 0} \int \sup_{r \geq 0} f(t+s+r,X_{t+s+r})dP(t+s,X_{t+s})dP(t,x) \]
\[ = \sup_{s \geq 0} \int \sup_{r \geq s} f(t+r,X_{t+r})dP(t,x) \]
\[ \leq \int \sup_{s \geq 0} f(t+s,X_{t+s})dP(t,x) . \]

Inequality (1.16) now follows from \( f_n \uparrow f_\infty \).

Next, fix \( t < t_0 < r \), let \( B_r = \sigma(X_s, s \leq r) \), and let \( E(t,x)\{ \cdot | B_r \} \)
be a conditional expectation relative to the measure \( P(t,x) \) and the
field \( B_r \). Then, using (1.16):

(1.16a) \[ f_\infty(r,X_r) \leq E(t,x)\{ \sup_{s \geq r} f(s,X_s) | B_r \} \]
\[ \leq E(t,x)\{ \sup_{s \geq t_0} f(s,X_s) | B_r \} . \]

Letting \( r \to \infty \), we have \( P(t,x) \) a.e.:

(1.16b) \[ \lim_{r \to \infty} f_\infty(r,X_r) \leq \lim_{r \to \infty} E(t,x)\{ \sup_{s \geq t_0} f(s,X_s) | B_r \} = \sup_{s \geq t_0} f(s,X_s) . \]

Letting \( t_0 \to \infty \) and using continuity of \( f \) and (1.3):

\[ \lim_{r \to \infty} f_\infty(r,X_r) \leq f(\infty,X_\infty) \quad P(t,x) \text{ a.e.} \]

Since \( f_\infty(r,X_r) \geq f(r,X_r) \), the other direction is obvious.

Theorem (1.1)

\[ f_\infty(t,x) = \int f_\infty(T_\infty,X_{T_\infty})dP(t,x), \text{ all } (t,x) \in [0,\infty) \times \mathbb{R} . \]

Proof: Dynkin \[ \] shows the following. If \( \epsilon > 0 \) and \( \Gamma_\epsilon = \{(t,x) : f_\infty(t,x) \leq f(t,x) + \epsilon \} \), then with \( T_\epsilon \) the hitting time for \( \Gamma_\epsilon \) and
with \( f \) bounded:
\[ f_\infty(t,x) = \int f_\infty(T_{\zeta}, X_{t_{\zeta}}) d\mu(t,x). \]

It is clear that \( T_{\varepsilon} \) is closed, that \( T_{\varepsilon} + T_{\infty} \) and that \( T_{\varepsilon} \uparrow T_{\infty} \) so that by our quasi-left continuity assumptions \( X_{t_{\varepsilon}} \to X_{t_{\infty}} \). Further,

\[ f_\infty(t,x) \leq \int f(T_{\varepsilon}, X_{t_{\varepsilon}}) d\mu(t,x) + \varepsilon \]

Letting \( \varepsilon \to 0 \) and using the quasi-left continuity of \( X \), and continuity and boundedness of \( f \), we have

\[ f_\infty(t,x) \leq \int f(T_{\infty}, X_{t_{\infty}}) d\mu(t,x). \]

Since \( f \leq f_\infty \) and since \( f_\infty \) is excessive and bounded, we've established our result in the bounded case. We now consider the unbounded case, subject as usual to (1.4). Set, for each \( a > 0 \):

\[ f_a = \min(f, a) \]

\[ \tilde{f}_a = \text{least excessive majorant of } f_a \]

\[ \Gamma_a = \{ f_a = \tilde{f}_a \} \]

\[ \tilde{\Gamma}_a = \{ f \geq \tilde{f}_a \} \]

\[ T_a = \text{hitting time for } \Gamma_a \]

\[ N_a = \text{hitting time for } \tilde{\Gamma}_a \]

Note that \( \Gamma_a \subseteq \tilde{\Gamma}_a \), that \( N_a \leq T_a \) and the \( b \geq a \) implies \( \tilde{f}_b \geq \tilde{f}_a \) by properties of least excessive majorants. Let \( \tilde{f}_\infty = \tilde{f}_a \) as \( a \to \infty \).

Clearly \( \tilde{f}_\infty \) is excessive and \( \tilde{f}_\infty \geq f \), therefore \( \tilde{f}_\infty \geq f_\infty \). On the other hand, \( f_\infty \geq f_a \), hence \( f_\infty \geq \tilde{f}_a \), from which \( f_\infty = \tilde{f}_\infty \), so that
for if we set \( N_0 = t \), then \( N_0 \leq T_0 \) while

\[
N_0 \not\rightarrow T_0, \quad \text{for if we set} \ N_0 = t < N_0, \ \text{then} \ N_0 \leq T_0.
\]

By lower semi-continuity of \( \tilde{f}_b \). But then

\[
f(N_0, X_{N_0}) \geq \lim_{a \to \infty} \tilde{f}_b(N_0, X_{N_0}) = f_{\infty}(N_0, X_{N_0})
\]

which implies \( (N_0, X_{N_0}) \in \Gamma_0 \), thus \( N_0 \geq T_0 \).

Next,

\[
\int f(N_a, X_{N_a})dP(t,x) \geq \int \tilde{f}_a(N_a, X_{N_a})dP(t,x) \geq \int \tilde{f}_a(T_a, X_{T_a})dP(t,x)
\]

since \( \tilde{f}_a \) is excessive and \( T_a \geq N_a \).

Since \( \tilde{f}_a \) is bounded, we have by Dynkin's result,

\[
\int \tilde{f}_a(T_a, X_{T_a})dP(t,x) = \tilde{f}_a(t,x)
\]

so that

\[
\int f(N_a, X_{N_a})dP(t,x) \geq \tilde{f}_a(t,x).
\]

Now, (1.4) allows us to use Lebesgue dominated convergence

\[
\int f(T_0, X_{T_0})dP(t,x) = \lim_{a \to \infty} \int f(N_a, X_{N_a})dP(t,x) = \lim_{a \to \infty} \int \tilde{f}_a(t,x) \geq \tilde{f}_a(t,x) = f_{\infty}(t,x).
\]

Since \( f_{\infty} \geq f \) and \( f_{\infty} \) is excessive, the result follows. Q.E.D.
Theorem (1.2)

\[ F(t,x) = f_{\infty}(t,x) = \int f(T_\infty, X_{T_\infty})dP(t,x), \quad \text{all } (t,x) \in [0, \infty) \times \mathbb{R} \]

Proof:

\[
F(t,x) = \sup_{t \geq T} \int f(T, X_T)dP(t,x) \geq \int f(T_\infty, X_{T_\infty})dP(t,x)
\]

\[
= \int f_{\infty}(T_\infty, X_{T_\infty})dP(t,x) = f_{\infty}(t,x)
\]

\[
\geq \sup_{t \geq T} \int f(T, X_T)dP(t,x)
\]

\[
\geq \sup_{T \geq t} \int f(T, X_T)dP(t,x)
\]

\[
= F(t,x) \quad \text{Q.E.D.}
\]

A Generalization

Consider the following situations:

(I) To determine \( F, T_\infty \) where \( N = \{N_t, \ t \geq 0\} \) is pure jump, where \( g \) is continuous, non-negative, and where, for \( A > 0 \)

\[
F(t,x) = \sup_{T \geq t} \int (g(T, N_T) - AN_T)dP(t,x)
\]

(II) To determine \( F, T_\infty \) where \( X = \{X_t, \ t \geq 0\} \) has continuous paths, where \( g \) is continuous, non-negative, and where for \( A > 0 \):

\[
F(t,x) = \sup_{T \geq t} \int (g(T, X_T) - AT)dP(t,x)
\]

In both cases we have an added cost factor which removes the integrand from the positive and, in a sense, bounded, type to which our previous theory applied. Since situations like (I), (II) are very natural, we will extend our results to include a broad class of such problems.
(1.17) Definition

We will call a process \( X \) which satisfies (1.1), (1.2), (1.3) extended standard. Dynkin [Markov Processes I, 104] defines as standard those processes \( X \) which satisfy (1.1), (1.2) and are quasi-left continuous. We will need a result from Dynkin:

(1.18) Result [Dynkin, Markov Processes I, Theorem 10.3]

Let \( X \) be a standard, \( G \) be open, \( T_G \) be the exit time from \( G \). Then \( \tilde{X} = \{X_{t \wedge T_G}, t \geq 0\} \) is standard.

The replacement of standard by extended standard leaves this result valid. We remark also that the statement of Theorem (10.3) in Dynkin is somewhat finer, involving measure theoretic conditions; however, Dynkin shows that there can always be met if \( X \) is standard.

(1.19) Definitions

Let \( f: [0,\infty) \times [\infty,\infty) \rightarrow [\infty,\infty) \) be continuous. We say that \( f \) decreases moderately with \( X \) if there exist two increasing sequences of positive constants \( \{a_n\}, \{b_n\} \) where \( a_n \nearrow \infty, a_n < b_n < a_{n+1} \), and

(a) If \( A_n = \{f > -a_n\} \), then \( [0,\infty) \times (\infty,\infty) \subset \bigcup_n A_n \)

(b) If \( \sigma_n \) is the exit time from \( A_n \), then \( f(\sigma_n, X_{\sigma_n}) \geq -b_n \).

The idea behind this definition is that although \( f(t, X_t) \rightarrow -\infty \) as \( t \rightarrow \infty \) is possible, this convergence doesn't take place too rapidly.

Theorem (1.3). Let \( X \) be extended standard and let \( f \) decrease moderately with \( X \). Suppose for all finite \( (t,x) \):

(A) \( \int |f(t+s, X_{t+s})| dP(t,x) < \infty \), all \( s \geq 0 \)
(B) \[ \int \sup_{s \geq 0} f^+(t+s, X_{t+s}) dP(t,x) < \infty. \]

(C) For all sequences \( \tau_n \) of increasing stopping times, \( t \leq \tau_n \neq \tau \), we have

\[ \int f(\tau_n, X_{\tau_n}) dP(t,x) + \int f(\tau_n, X_{\tau}) dP(t,x). \]

Let

\[ F(t,x) = \sup_{T \geq t} \int f(T, X_T) dP(t,x) \]

\( \Gamma_\infty = \{ f = F \} \)

\( T_\infty = \text{hitting time of } \Gamma \)

Then \( \Gamma_\infty \) is closed, \( T_\infty \) is a well-defined, possibly extended hitting time, and

\[ F(t,x) = \int f(T_\infty, X_{T_\infty}) dP(t,x) \]

**Proof:** Let \( A_n = \{ f > -a_n \} \) where, if \( \sigma_n \) is the exit time from \( A_n \), then the closure \( B_n \) of \( A_n \) union its exit points \( (\sigma_n, X_{\sigma_n}) \) satisfies \( B_n \subset A_{n+1} \). By Dynkin's theorem \( X^{(n)} = \{ X_{t \wedge \sigma_n} \} \) is extended standard when confined to \( B_n \). Since \( f \) restricted to \( B_n \) is bounded below by \( a_{n+1} \), we can use Theorem (1.2) as follows:

Let \( (t,x) \in B_n \); let

\[ F_n(t,x) = \sup_{T \in [t, \sigma_n]} \int f(T, X_T) dP(t,x) \]

\( \Gamma_n = \{ (t,x) \in B_n, f(t,x) = F_n(t,x) \} \)

\( T_n = \text{hitting time of } \Gamma_n \) for paths beginning in \( B_n \).
Then $T_n$ is closed in $B_n$, $T_n$ is a well-defined hitting time with $T_n \leq \sigma_n$, $F_n$ is lower semi-continuous on $B_n$, and is the least excessive majorant of $f$ for the process $X^{(n)}$ restricted to $B_n$, and

$$F_n(t,x) = \int f(T_n,T_n)dp_n(t,x), \text{ all } (t,x) \in B_n.$$  

Note that $F_n$ increases on $A_{n_0}$, all $n \geq n_0$, so that $F_\infty = \limsup F_n$ is well-defined and lower semi-continuous on $[0,\infty) \times (-\infty,\infty)$. Now for any finite $(t,x)$ and any stopping time $T$ beginning at $(t,x)$, (C) demands that

$$(1.20) \int f(T,X_T)dp(t,x) = \lim_{n\to\infty} \int f(T_\sigma_n,X_{T\wedge \sigma_n})dp(t,x)$$

But then we have $F_\infty = F$, for (1.20) shows $F \leq F_\infty$ while the definition of $F_n$ demands $F_\infty \leq F$.

We want now to be able to define $T_\infty$ as the hitting time of $\{F = f\}$, and since we want this definition to make sense at infinity, we'll establish that

$$(1.21) P(t,x)\left\{ \lim_{s \to \infty} F(s,X_s) = f(\sigma,X_\sigma) \right\} = 1, \text{ all finite } (t,x)$$

We refer the reader back to the proof of (1.15), noting here only that (1.16a) holds by reducing to $F_n$ and then passing to limits, while (1.16b) holds from the fact that $\sup f(s,X_s)$ is bounded above by $f(t_0,X_{t_0})$, and below by $\sup f(s,X_s)$ and below by $f(t_0,X_{t_0})$; then we use (A),(B). Thus, $T_\infty$ is well-defined since $F(\sigma,X_\sigma)$ is well-defined.

Let's next consider the times $T_n$. Clearly, for $n \geq n_0$ and on paths starting in $A_{n_0}$, $T_n$ is increasing in $n$, thus $T_n \uparrow T_\ast$, and our proof will be completed if we can show that $T_\ast = T_\infty$, for, by (C):
Fix finite \((t,x) \in A\). Now \(F(t,x) > -\infty\), thus

\[
P_{(t,x)}(\{f(T^*_n, X_{T^*_n}) = -\infty\}) = 0,
\]
so that for almost all paths beginning at \((t,x)\), there exists \(n\), possibly dependent on the path, with \((T^*_n, X_{T^*_n}) \in A_n\). Then by extended quasi-left continuity, there exists \(m\) such that the entire path, \(\{(s, X_s): s \in [t, T^*_m]\} \subseteq A_m\). But then

\[
f(T^*_n, X_{T^*_n}) = \lim_{n \to \infty} f(T_n, X_{T_n}) \geq \lim_{n} F_n(T_n, X_{T_n})
\]

\[
\geq \lim_{n} F_m(T_n, X_{T_n}) \geq F_m(T^*_m, X_{T^*_m})
\]

\[
\geq F(T^*_m, X_{T^*_m})
\]

where the last inequality depends on the lower semi-continuity of \(F_m\) and where the next to last inequality depends on the monotonicity of \(F_n\) in \(n\) on \(A_m\). Thus, \(T^*_n \leq T^*_m\). On the other hand, \(T_n \leq T^*_m\), so \(T^*_n \leq T^*_m\). Q.E.D.

\((1.22)\) Remarks on Discrete Time Optimal Stopping

The theory developed above holds for Markov processes that move on a discrete time lattice \(\{kr, k = 0, 1, 2, \ldots\}, r > 0\) provided the discrete analogues of hypotheses involving transitions \(p(t,x,dy)\) are used. In point of fact, the theory falls out much more simply and with fewer hypotheses and we will not develop it here.
Applications

Let $X = \{X_t, t \geq 0\}$ be a standard process which is homogeneous in time, i.e., $P(t, x) = P_x$. This assumption will hold unless we specify otherwise. Assume further that for all $\lambda > 0$:

\[(a) \quad \sup_{s \geq 0} e^{-\lambda(t+s)} |X_{t+s}| dP(t, x) < \infty \]

\[(b) \quad P(t, x) \left\{ \lim_{s \to \infty} e^{-\lambda(t+s)}X_{t+s} = 0 \right\} = 1 \]

Then, we have from Theorem (1.3) in conjunction with Remarks (1.6) that there exists $\Gamma_\infty$, a closed set, and its hitting time, $T_\infty$, such that

\[(2.2) \quad F(t, x) = \sup_{T \geq t} \int e^{-\lambda T} X_T dP(t, x) = \int e^{-\lambda T} X_{T_\infty} dP(t, x) \]

It will be convenient to use a different notation for stopping times. Since $T_\infty$ is a hitting time, we can restrict our stopping time considerations exclusively to hitting times, and given any closed $\Gamma$, we will let $\tau$ be the time it takes to enter $\Gamma$, starting from $(t, x)$, and $X_\tau$ the increment in distance required. We then re-write (2.2) in the form

\[ F(t, x) = \sup \int e^{-\lambda(t+\tau)} (x + X_\tau) dP_x = \int e^{-\lambda(t+\tau_\infty)} (x + X_{\tau_\infty}) dP_x \]

where $\tau_\infty$ is the time it takes to travel from $(t, x)$ into $\Gamma_\infty$. 
Now $\Gamma_\infty$ is specified by

\begin{equation}
(t,x) \in \Gamma_\infty \iff e^{-\lambda t} x \geq \sup_{\tau > 0} \int e^{-\lambda (t+\tau)} (x+x_\tau) dP_x
\end{equation}

which we write in the equivalent form

\begin{equation}
(t,x) \in \Gamma_\infty \iff x \geq H(x) = \begin{cases} 
\sup_{\tau > 0} \int e^{-\lambda \tau} x_\tau dP_x & \text{if } x \neq 0 \\
1 - \int e^{-\lambda \tau} dP_x & \text{if } x = 0
\end{cases}
\end{equation}

In most of the analysis which follows we will consider $x \neq 0$, the case $x = 0$ being either similar or irrelevant.

The description (2.5) demands that $\Gamma_\infty$ have form

\begin{equation}
\Gamma_\infty = \{(t,x) : x \geq H(x) \text{ or } t = \infty\}
\end{equation}

If we use $\tau_0$ for the hitting time of $\Gamma_\infty$ starting from $(t,x)$, we have

\begin{equation}
H(x) = \frac{\int e^{-\lambda \tau_0} x_\tau_0 dP_x}{1 - \int e^{-\lambda \tau_0} dP_x} \quad x \neq 0
\end{equation}

\begin{equation}
\text{Remarks}
\end{equation}

We introduce some definitions and notation which will facilitate the statement and proof of the next theorem.
(I) Let \( a < x < b \) be in the state space of \( X \). Let \( \tau \) be the exit time from \((a,b)\), starting from \( x \), and let \( x_\tau \) be the distance travelled from \( x \) at exit time. If, along all paths where \( \tau < \infty \) we have \( x + x_\tau \in \{a,b\} \), we will say that \( X \) doesn't skip states.

(II) Two states \( a < b \) in the state space of \( X \) will be called non-contiguous if there exists \( c \) in the state space of \( X \) and \( c \in (a,b) \).

Theorem (2.1)

Let \( X = \{X_t, t \geq 0\} \) be a standard process which satisfies (2.1) and which doesn't skip states. Suppose further that for all \( 0 \leq a < b \leq \infty \) where \( a \) and \( b \) are non-contiguous, there exists \( x \in (a,b) \) where, if \( \tau \) is the exit time from \((a,b)\), starting at \( x \):

\[
\begin{align*}
\{ & P \{ \tau < \infty \} = 1 \\
& x \geq \int e^{-\lambda \tau} (x + x_\tau) dP_x
\end{align*}
\]

Then there exists a finite \( x_0 \geq 0 \) such that

\[
\Gamma_\infty = \{ (t,x) : x \geq x_0 \ or \ t = \infty \}
\]

We have

\[
F(t,x) = \begin{cases} 
  e^{-\lambda t} x & \text{if } x \geq x_0 \\
  e^{-\lambda t} x_0 \int e^{-\lambda \tau_0} dP_x & \text{if } x < x_0
\end{cases}
\]

where \( \tau_0 \) is the hitting time for \( x_0 \), starting from \( x \).
Proof:

Note first that \( x < 0 \) implies \( (t, x) \notin \Gamma_\infty \), for from (2.1), we can always do better than the negative value \( e^{-\lambda t} x \) simply by continuing on to \( X_\infty \). Thus, if \( \Gamma_\infty \) is not of the proposed form, then (2.6) demands the existence of non-contiguous \( 0 \leq a < b \leq \infty \) with \( a \geq H(a), \ b \geq H(b) \) but \( x < H(x) \), all \( x \in (a, b) \). We note next that for all such \( x \), the time, \( \tau_0 \), to travel from \( (t, x) \) into \( \Gamma_\infty \) is the same as the time \( \tau \), to exit \( (a, b) \), starting at \( x \), for \( X \) does not skip states. From (2.7), for \( x \in (a, b) \)

\[
x < H(x) = \frac{\int e^{-\lambda \tau_0} x_{\tau_0} dP_x}{1 - \int e^{-\lambda \tau_0} dP_x}
\]

On the other hand, for some \( x \in (a, b) \), and using (2.9) and the equivalence \( \tau = \tau_0 \):

\[
x \geq \int e^{-\lambda \tau_0} (x + x_{\tau_0}) dP_x
\]

so that we reach a contradiction.

Thus \( \Gamma_\infty = \{ (t, x) : x \geq x_0 \ \text{or} \ t = \infty \} \) and it remains to show that \( x_0 \) is finite. If \( x_0 = \infty \), then \( \Gamma_\infty = \{ (t, \infty) : t \geq 0 \} \) and using (2.1) we would have \( F(t, x) = 0 \), all finite \( t \) with \( (t, x) \notin \Gamma_\infty \), leading to the contradiction \( F(t, x) < e^{\lambda t} x \) for \( (t, x) \notin \Gamma_\infty, \ x > 0 \).

(2.10) Remarks

The class of processes \( X \) for which (2.9) holds is probably quite large. It includes, for instance, super martingales for which
the first half of (2.9) holds, for we have, with $0 \leq a < x < b$, and $\tau$
the exit time from $(a,b)$:

$$x \geq \int (x+x_t) dP_x > \int e^{-\lambda \tau (x+x_t)} dP_x.$$  

An Application to Diffusions

Let $X = \{X_t, t \geq 0\}$ be a diffusion.

For $0 \leq x \leq y < \infty$ we assume

(I) $P_x \{\tau_y < \infty\} = 1$, $\tau_y$ the hitting time for $y$

(II) $\int e^{-\lambda \tau} dP_x$ is continuous in $x$, all $\lambda > 0$

(III) $\frac{d}{dx} \int e^{-\lambda \tau} dP_x$ is continuous in $x$, all $x < y$, $\lambda > 0$

(IV) $H(\lambda, y) = \lim_{x \uparrow y} \frac{d}{dx} \int e^{-\lambda \tau} dP_x$ exists and is continuous in $y$

Theorem (2.3)

Let $X$ be a diffusion satisfying (2.1), (2.9) and (I) through (IV) above. Assume $xH(\lambda, x) = 1$ has a unique non-negative solution, call it $x_0$. Then

(2.11) $\Gamma_\infty = \{t, x): x \geq x_0 \text{ or } t = \infty\}$

Further, $x_0 > 0$.

Proof:

First $x_0 > 0$, for if $(0, t) \in \Gamma_\infty$ then for all $\tau$ we would have

$$0 \geq \int e^{-\lambda \tau X_t} dP_0,$$
but using (I) with \( \tau = \tau_y \), we would then have

\[ 0 \geq \int y e^{-\lambda \tau_y} dP_0 > 0 , \]

a contradiction.

If \( x < x_0 \), and if \( \tau_0 \) is the time to hit \( x_0 \), starting from \( x \), then, from (2.5):

\[
x < (x_0 - x) \int e^{-\lambda \tau_0} dP_x \\
\frac{1 - \int e^{-\lambda \tau_0} dP_x}{1 - \int e^{-\lambda \tau_0} dP_x}
\]  \hspace{1cm} (2.12)

Using the Mean Value Theorem on the denominator, there exists \( z \in (x, x_0) \) with

\[ 1 - \int e^{-\lambda \tau_0} dP_x = (x_0 - x) \frac{d}{dz} \int e^{-\lambda \tau_0} dP_z , \]

so that, (2.11) becomes

\[
x < \int e^{-\lambda \tau_0} dP_x \\
\frac{d}{dz} \int e^{-\lambda \tau_0} dP_z
\]

Now let \( x \to x_0 \) so that \( z \to x_0 \). Our hypothesis (III, IV) leads to

\[ x_0 \leq \frac{1}{H(\lambda, x_0)} \]

On the other hand, if \( x_0 \leq x \leq x_1 \), and if \( \tau_1 \) is the time to reach \( x_1 \), starting from \( x \), then since the best policy at \( x \) is to stop at
\[ e^{-\lambda t} x > \int_0^{\tau_1} e^{-\lambda (t+t_1)} x_1 \, dp_x, \quad \text{i.e.,} \]

\[
x_1 \geq (x_1 - x) \frac{\int e^{-\lambda t_0} \, dp_x}{1 - \int e^{-\lambda t_1} \, dp_x}
\]

Repeating our reasoning above, we have

\[
x_1 \geq \frac{1}{H(\lambda, x_1)}
\]

Q.E.D.

(2.13) **Example**

The stationary Ornstein-Uhlenbeck process has the representation

\[
X_t = e^{-at} \left( X_0 + \sqrt{2ab} \int_0^t e^as \, dw_s \right)
\]

where \( W \) is standard brownian motion, \( X_0 \) is \( n(0,b) \), and \( W \) and \( X_0 \) are independent. The transform

\[
\psi(x, x_0) = \int e^{-\lambda t_0} \, dp_x, \quad x < x_0
\]

satisfies

\[
ab\psi'' - ax\psi' = \lambda \psi, \quad \psi(x_0) = 1.
\]

This equation is solved in Darling-Siegert [ ]. When \( a = b = \lambda = 1 \), the solution takes the relatively simple form
The process $X$ can be shown to satisfy the hypotheses of Theorem (2.2), consequently the optimal $x_0$ is the unique value for which

$$x_0 \lim_{x \to x_0} \frac{d}{dx} \left( \int_0^\infty e^{-\frac{1}{2} t^2} dt \right) = 1$$

Straightforward calculations reduce this to the equation

$$x_0^2 + x_0 e^{-\frac{1}{2} x_0^2} = 1$$

for which the solution is $x_0 \approx 0.839$, agreeing with Taylor [ ].

**Theorem (2.3)**

Let $X = \{X_t, t \geq 0\}$ be a standard process satisfying (2.1) and having independent stationary increments. Assume for $x < y$ that $P_x \{\tau_y < \infty\} > 0$. Then there exists finite $x_0 > 0$ with $\Gamma_\infty = \{(t,x): x \geq x_0 \text{ or } t = \infty\}$. 
Proof:

Since \( P_x \{ \tau_y < \infty \} > 0 \), we see that \((t,x) \in \Gamma_\infty \Rightarrow x > 0 \). Thus \((t,x) \in \Gamma_\infty \) if and only if

\[
(x, \sup_{\tau > 0} \frac{e^{-\lambda \tau} x_\tau dP_x}{1 - e^{-\lambda \tau} dP_x})
\]

(2.14)

The assumption that \( X \) has stationary independent increments implies that the right side above is a constant not dependent on \( x \). Let this constant be \( x_0 \). As in the last theorem, \( x_0 \) is necessarily finite.

(2.15) Applications

(1) Let \( X = \{ X_t, t \geq 0 \} \) be Brownian Motion with drift \( m \) variance \( \sigma^2 \). Then \( X \) has continuous paths and stationary independent increments, so that if \( \tau_0 \) is the hitting time of \( x_0 \), starting from \( x < x_0 \), we have

\[
\int e^{-\lambda \tau_0} dP_x = e^{-(x_0 - x)G(\lambda)}
\]

where

\[
G(\lambda) = \frac{\sqrt{M^2 + 2\lambda \sigma^2} - M}{\sigma^2}
\]

from which it follows immediately that
\[ x_0 = \frac{\sigma^2}{\sqrt{M^2 + 2\lambda_0^2}} - M \]

and

\[
F(t, x) = \begin{cases} 
\exp(-\lambda t x) & \text{if } x \leq x_0 \\
x_0 \exp(-\lambda t (x_0 - x)G(\lambda)) & \text{if } x > x_0 
\end{cases}
\]

This agrees with Taylor [ ].

(2) Let \( X = \{X_t, t \geq 0\} \) be a stationary independent increment pure jump process with state space the non-negative integers and with all jumps positive. More specifically, if \( T \) is generic for jump times, and if \( x, z \) are non-negative integers, with \( z > 0 \), then there exists fixed \( a > 0 \) and some fixed probability \( \theta \) on the positive integers with

\[
P_x\{T > t\} = e^{-at}
\]

\[
P_x\{X_T = x + z\} = \theta(z)
\]

We will assume \( \sum z^2 \theta(z) < \infty \), from which it follows fairly easily that conditions (1.7) are met and Theorems (1.2), (2.3) are available.

Letting \( \tau_0 \) be the time to exit \([0, x_0]\) where \( x_0 \) is the optimal stopping value, we see that

\[
x_0 - 1 < \int e^{-\lambda_0 \tau_0} \frac{d\tau_0}{x_0 - 1} \frac{d\tau_0}{1 - \int e^{-\lambda_0 \tau_0} \frac{d\tau_0}{x_0 - 1}}
\]
However, \( \tau_0 = T \), the jump time from starting point \( x_0 - 1 \), so that we have

\[
x_0 - 1 < \frac{\int e^{-\lambda T} z_T dp_{x_0 - 1}}{1 - \int e^{-\lambda T} dp_{x_0 - 1}}
\]

(2.16)

Likewise, since \( x_0 \) is optimal:

\[
x_0 \geq \frac{\int e^{-\lambda t} z_T dp_{x_0}}{1 - \int e^{-\lambda T} dp_{x_0}}
\]

(2.17)

where again we are using \( T \) generically as the jump time from any given starting point. One calculates that the right side of (2.17) is \( \frac{a}{\lambda} \sum z_0(z) \), so that \( x_0 \) is the smallest integer with

\[
x_0 \geq \frac{a}{\lambda} \sum z_0(z)
\]

(2.18)

For the case \( \theta(1) = 1 \), we have as optimal payoff:

\[
F(t,x) = \begin{cases} 
  e^{-\lambda t} x & \text{if } x \leq x_0 \\
  x_0 \left( \frac{a}{a+\lambda} \right)^{x_0-x} e^{-\lambda t} & \text{if } x < x_0
\end{cases}
\]

(2.19)

and this agrees with Taylor [ ].
Let $X = \{X_t, \ t \geq 0\}$ be brownian motion with non-negative drift. We assume $P\{X_0 = 0\} = 1$. Let $c: [0, \infty) \to (0, \infty)$ be strictly decreasing, continuous. We want to determine $\Gamma_\infty, T_\infty$ where

$$F(t,x) = \sup_{T \geq t} \int c(T)X_T dP(t,x) = \int c(T_\infty)X_{T_\infty} dP(t,x)$$

We will be able to use the characterization of Theorem (1.3) for $\Gamma_\infty$ provided, for instance

$$\begin{cases} P(0,0) \left\{ \lim_{t \to \infty} c(t)X_t \text{ exists and is finite} \right\} = 1 \\ \int \sup_{t \geq 0} c(t) |X_t| dP(0,0) < \infty. \end{cases}$$

The restriction to $P(0,0)$ in (3.2) is allowable since $X$ has stationary independent increments. Conditions (3.2) will hold if, for instance, in analogy with Proposition (1.1), we have

$$\lim_{t \to \infty} tc(t) = M < \infty$$

We restrict $c$ so that not only (3.3) holds but also

$$c', \text{ the derivative of } c, \text{ is continuous with strictly decreasing absolute value, and}$$

$$\left| \frac{c'(t)}{c(t)} \right| \text{ is non-increasing in } t.$$
\[ \Gamma_\infty = \{(t,x) : x \geq G(t) \text{ or } t = \infty\} \]

where

(I) \( G \) is non-decreasing

(II) For all \( s > 0 \), all \( 0 \leq r \leq t \leq s \):

\[ G(t) - G(r) \leq \frac{M}{B(s)} (t-r) \]

where \( M \) is a universal constant and \( B : [0, \infty) \to (0, \infty) \).

**Proof:** \((t,x) \in \Gamma_\infty\) if and only if

\[ c(t)x \geq \sup_{\tau > 0} \int c(t+\tau)(x+x_\tau)dP(t,x) = F(t,x) \]

Given any \( x < y \), with \( T \) the time to travel from \( x \) to \( y \), we have

\[ P_x(T < \infty) = 1 \]

therefore \( F(t,x) > 0 \), hence \((t,x) \in \Gamma_\infty\) implies \( x > 0 \), and we can re-write (3.5) as

\[ (t,x) \in \Gamma_\infty \iff x \geq \sup_{\tau > 0} \int \frac{c(t+\tau)}{c(t)} x_\tau dP(t,x) \]

Now we know that \( \tau \) can be restricted to hitting times of closed sets, \( \Gamma \), and that the fact that \( X \) has stationary independent increments gives \( \tau \), the hitting time of \( \Gamma \), starting from \((t,x)\), the same stochastic structure is \( \tau' \), the hitting time of \( \Gamma - (t,x) \), starting from \((0,0)\).

This allows us to re-write (3.6) as

\[ (t,x) \in \Gamma_\infty \iff x \geq \frac{\int \frac{c(t+\tau)}{c(t)} x_\tau dP(0,0)}{1 - \int \frac{c(t+\tau)}{c(t)} dP(0,0)} \]

Further, (3.4) implies
(3.8) \[ s \geq t \Rightarrow \frac{c(s+r)}{c(s)} \geq \frac{c(t+r)}{c(t)} \]

so that \( G(t) \) is non-decreasing in \( t \). It is also clear that \( G(0) > 0 \).

Next, if \( 0 < x < G(t) \), the reasoning which led from (3.5) to (3.7) allows the following definition of \( G \):

(3.9) \[
G(t) = \frac{\int \frac{c(t+\tau_0)}{c(t)} X \tau_0 dP(0,0)}{1 - \int \frac{c(t+\tau_0)}{c(t)} dP(0,0)}
\]

where \( \tau_0 \) is the time to reach \( G \), starting from \( (t,x) \). Fixing \( x = \frac{G(0)}{2} \), we can re-define \( G \) with

(3.10) \[
G(t) = \sup_{\tau \geq \tau_1} \frac{\int \frac{c(t+\tau)}{c(t)} X \tau dP(0,0)}{1 - \int \frac{c(t+\tau)}{c(t)} dP(0,0)}
\]

where \( \tau_1 \) is the time to travel from zero to \( \frac{G(0)}{2} \). That is, we are free to define \( G \) through stopping times which are strictly bounded away from zero so that the denominator above doesn't get too small. Let's now prove (II). We set

(3.11) \[
G(t,\tau) = \frac{\int \frac{c(t+\tau)}{c(t)} X dP(0,0)}{1 - \int \frac{c(t+\tau)}{c(t)} dP(0,0)}
\]

We first get bounds on \( G(s,\tau) - G(t,\tau) \), \( s \geq t \).
By straightforward calculations
\[
\left| \int \frac{c(t+\tau)}{c(t)} X_\tau dP(0,0) - \int \frac{c(s+\tau)}{c(s)} X_\tau dP(0,0) \right|
\leq \int \sup_{r \geq 0} \left| \frac{c(s+r)}{c(s)} - \frac{c(t+r)}{c(t)} \right| |X_r| dP(0,0)
\leq 2 \left| \frac{c'(0)}{c(0)} \right| \cdot \frac{1}{c(s)} \cdot |t-s| \int \sup_{r \geq 0} c(r) |X_r| dP(0,0)
= M \cdot \frac{|t-s|}{c(s)} \text{ for some universal } M > 0.
\]
Likewise
\[
\left| \int \frac{c(t+\tau)}{c(t)} dP(0,0) - \int \frac{c(s+\tau)}{c(s)} dP(0,0) \right| \leq \frac{M}{c(s)} |t-s|
\]
where we again use $M$ for a generic universal constant, not necessarily the same from case to case.

But then, for all $\tau \geq \tau_1$ where $\tau_1$ is as in (3.10): $G(s,\tau) - G(t,\tau) \leq \frac{M}{c^2(s)} \cdot \frac{|s-t|}{B(\tau,s)}$ where $B(\tau,s) = \left(1 - \int \frac{c(s+\tau)}{c(s)} dP(0,0)\right)^2$, and the latter quantity, with $\tau \geq \tau_1$, is bounded away from zero. Finally, given $\epsilon > 0$, $t \leq s \leq s_0$ choose $\tau \geq \tau_1$ so that
\[
G(s) \leq G(s,\tau) + \epsilon.
\]
We then have
\[
G(s) \leq G(t,\tau) + \epsilon + \frac{M \cdot |s-t|}{c^2(s)B(\tau,s)}
\leq G(t) + \epsilon + \frac{M \cdot |s-t|}{B(s_0)}
\]
where $B(s_0) = \inf_{\tau \geq \tau_1} c^2(s_0)B(\tau,s_0)$. Q.E.D.
(3.12) Remarks

(I) Let $X$ be standard brownian motion; let $c(t) = \frac{1}{(A+Bt)^{1/2}}$, $r > 1/2$, $A > 0$, $B > 0$. Then

$$G(t) = \sup_{\tau > 0} (A+Bt)^{r} \frac{\int X_{\tau} \frac{dP}{(A+B(t+\tau))^{1/2}}} {1 - (A+Bt)^{r} \int \frac{1}{(A+B(t+\tau))^{1/2}} dP(0,0)}.$$ 

Using the fact that $X_{\tau} \overset{d}{=} aX_{\tau/a^{2}}$ for $X$ standard brownian motion, setting $a = \frac{\sqrt{A+Bt}}{B}$ and substituting in the integral above, we have

$$G(t) = K_{T} \cdot \sqrt{\frac{A+Bt}{B}},$$

where

$$K_{T} = \sup_{\tau > 0} \frac{\int X_{\tau} \frac{dP}{(1+\tau)^{1/2}}} {1 - \int \frac{1}{(1+\tau)^{1/2}} dP(0,0)}.$$ 

We see then that

$$\Gamma_{\infty} = \left\{ (t,x) : x \geq K_{T} \cdot \sqrt{\frac{A+Bt}{B}} \text{ or } t = \infty \right\}.$$ 

The constant $K_{T}$ is determined by Walker [ ] and in special cases by Shepp [ ] and Taylor [ ]; we will not pursue this problem here.

(II) The last theorem, although restricted to Brownian motion with non-negative drift, can be extended, with minor changes in hypotheses and conclusions, to a large class of processes with stationary independent increments. Again, we will not develop these generalizations here.
Weak Convergence

For each $n$, let $X_n = \{X_n(t), t = m\phi(n), m = 0, 1, 2, \ldots\}$ be a discrete time Markov process whose time parameter lines on the lattice $\{m\phi(n)\}$ where $\phi(n+1)$ divides $\phi(n)$, and $\phi(n) \to 0$ as $n \to \infty$, for instance, we might have $\phi(n) = 2^{-n}$. We assume $X_n$ is governed by a transition $p_n$, i.e.,

\[(3.13) \quad p\{X_n(t+\phi(n)) \in dy | X_n(t) = x\} = p_n(x, dy).\]

Let $f: [0, \infty) \times \mathbb{R} \to [0, \infty)$ be continuous. We define

\[(3.14) \quad F_n(t, x) = \sup_{T \geq t} \mathbb{E}_{T} f(T, X_T) dP_n(t, x)\]

where it is to be understood that $T$ runs through stopping times on the lattice $\{m\phi(n)\}$ and where $p_n(t, x)$ is the probability on the process $X_n$ under initial state $(t, x)$. In order that extended times $T$ be available we'll assume always that

\[(3.15) \quad p_n(t, x) \{f(\infty, X_{\infty}) = 0\} = 1, \text{ all } (t, x), \text{ all } n.\]

For each $n$, extend the process $X_n$ to $[0, \infty)$ by the standard interpolation:

\[(3.16) \quad X_n(t) = \frac{((m+1)\phi(n) - t)}{\phi(n)} X_n(m\phi(n)) + \frac{(t - m\phi(n))}{\phi(n)} X_n((m+1)\phi(n))\]

for $t \in [m\phi(n), (m+1)\phi(n)]$. We will continue to use $X_n$ as notation for the interpolated process and we'll also use $p_n(t, x)$ for $X_n$ conditioned on $X_n(t) = x$. Where no ambiguity arises we'll write $F(t, x)$ for $F(t, x)$. Now assume there exists $X_\infty$, a standard continuous path Markov process with transition $p_\infty(t, x, dy)$ such that

\[X_n \xrightarrow{D} X_\infty \quad \text{as} \quad n \to \infty\]
where the intended convergence is weak convergence on \( C[0, \infty) \) under all initial \((t, x)\) in the state space of \( X_n \) for large \( n \), i.e., under all \((t, x)\) where \( t \) is on the \( \phi(n) \) lattice, all large \( n \). For treatments of weak convergence see Billingsley [ ] and Whitt [ ].

We'll assume that \( P_{(t, x)} \{ f(0, X_n(0)) = 0 \} = 1 \), all \((t, x)\) and that \( T_\infty \) exists where

\[
F_\infty(t, x) = \sup_{T \geq t} \int f(T, X_n(T)) dP_{(t, x)} = \int f(T, X_n(T)) dP_{(t, x)}
\]

and \( T_\infty \) is the hitting time of the closed set \( \Gamma_\infty = \{ f = F_\infty \} \), and \( F_\infty \) is the least excessive majorant under \( p_\infty \) of \( f \). Our objective is to relate \( F_\infty \) to \( F_n \) through \( T_\infty \) in a manner to be made precise below. We begin with some assumptions.

**Assumptions**

(3.18) For all \( n \leq \infty \) and for all \((t, x)\) in the space-time range of \( X_n \):

\[
P_{(t, x)} \left\{ \lim_{s \to \infty} f(s, X_n(s)) = 0 \right\} = 1.
\]

Thus, we interpret \( f(0, X_n(0)) = 0 \), as already indicated above.

(3.19) \( f(t, x) \) increases in \( x \), decreases in \( t \), and is uniformly continuous in \( t \).

(3.20) There exists \( G \in C[0, \infty) \) non-decreasing in \( t \) with

\[
\Gamma_\infty = \{ (t, x): x \geq G(t) \text{ or } t = \infty \}.
\]

(3.21) For all \((t, x)\):

\[
P_{(t, x)} \left\{ \bigcap_{\epsilon > 0} \bigcup_{0 < s < \epsilon} \left\{ X_n(T_\infty + s) > G(T_\infty + s) \right\} \right\} = 1
\]

conditional on \( \{ T_\infty < \infty \} \).
(3.22) For all large \( n \), \( F_\infty \) is excessive for discrete time \( X_n \), i.e.,

\[
F_\infty(m\phi(n),x) \geq \int F_\infty((m+1)\phi(n),y)p_n(x,dy) .
\]

(3.23) **Comments on the assumptions**

(I) The determination of \( f \) though non-decreasing, continuous \( G \) is motivated by Theorem (3.1).

(II) Assumption (3.21) prevents \( X_\infty \) from moving away from \( \Gamma_\infty \) once it hits \( \Gamma_\infty \). This is a rather weak assumption, true, for instance, when \( X_\infty \) is brownian motion and \( G \) has less than infinite slope everywhere — a proof is easily established by the local law of the iterated logarithm.

Let's now define \( T_n \) as the approximate hitting time of \( \Gamma_\infty \) for the discrete time process \( X_n \):

\[
T_n = \begin{cases} 
\text{minimal } m\phi(n) > X_n(t) \in \Gamma_\infty & \text{for } t \in ((m-1)\phi(n), m\phi(n)) \\
\infty & \text{if no such } m \text{ exists}.
\end{cases}
\]

Our main result is that \( T_n \) is approximately optimal for the problem defined in (3.14).

**Theorem (3.2).** Let \( X_n \overset{D}{\rightarrow} X_\infty \) under all initial \( (t,x) \) in the space-time range of \( X_n \), large \( n \). Assume (3.18) through (3.22). Then, for all such \( (t,x) \):

\[
F_\infty(t,x) = \lim_{n \to \infty} F_n(t,x) = \lim_{n \to \infty} \int f(T_n, X_n(T_n))dP(t,x) .
\]

**Proof:** By Shorokhod's theorem — see Billingsley [ ] — there exists a probability triple \((\Omega, B, P)\) supporting random elements \( Y_n : \Omega \to C[0,\infty), n \leq \infty \) such that
Since the matrix on $C[0,\infty)$ is uniform convergence on all closed intervals $[0,t_0]$, it is easily seen from assumptions (3.20) and (3.21) that if $V_n$ is the hitting time of $G(t)$ for the continuous time process $X_n$ and if $S_n$ is the corresponding hitting time for $Y_n$, then

\begin{equation}
V_n \overset{D}{=} S_n, \quad \text{all } n \leq \infty
\end{equation}

and

\begin{equation}
S_n \to S_\infty \quad \text{on } \{S_\infty > \infty\}.
\end{equation}

Clearly, (3.21) implies (3.27) by making hitting times $C[0,\infty)$-continuous where they occur. But then, by continuity of $f$ and $Y_n$, we have

\begin{equation}
f(S_n,Y_n(S_n)) \to f(S_\infty,Y_\infty(S_\infty)), \quad \text{all } \omega \in \Omega.
\end{equation}

Here we use (3.18) on $\{S_\infty = \infty\}$. Now (3.26) demands that

\begin{equation}
\int f(V_n,X_n(V_n))dP(t,x) = \int f(S_n,Y_n(S_n))dP, \quad \text{all } n \leq \infty
\end{equation}

and then Fatou's lemma applied to the right and interpreted for the left yields (note $V_\infty = T_\infty$):

\begin{equation}
\int f(T_\infty,X_\infty(T_\infty))dP(t,x) \leq \lim_{n} \frac{1}{n} \int f(V_n,X_n(V_n))dP(t,x).
\end{equation}

Now we want to replace $V_n$ by $T_n$ so that we can relate behaviour of discrete time $X_n$ to continuous time $X_\infty$. Since $|T_n-V_n| \leq \phi(n)$, and $\phi(n) \to 0$, the uniform continuity of $f$ in $t$ demands that for large $n$

\begin{equation}
f(V_n,X_n(V_n)) \leq f(T_n,X_n(V_n)) + \epsilon.
\end{equation}
Further, since $G$ is non-decreasing and $V_n$ is the hitting time for $G$, and $f$ is increasing in $x$:

\[(3.32) \quad f(T_n, X_n(V_n)) \leq f(T_n, X_n(T_n)) .\]

Consequently:

\[(3.33) \quad \int f(T_n, X_n(T_n)) dP(t,x) \leq \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t,x) .\]

Finally, assumption (3.22) and the properties of least excessive majorants gives $F_\infty \geq F_n$ so that, using (3.33)

\[(3.34) \quad F_\infty(t,x) = \int f(T_\infty, X_\infty(T_\infty)) dP(t,x) \leq \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t,x) \leq \lim_{n \to \infty} F_n(t,x) \leq F_\infty(t,x) .\]

Q.E.D.

A Related Result for $D[0,\infty)$

Let $\{X_n\}$ be a discrete time sequence of integer-valued processes where $X_n$ has time lattice $\{m\phi(n)\}$, and where $X_n$ has transitions governed by $p_n$ where, for all integer $x,y$:

\[P(X_n((m+1)\phi(n)) = y | X_n(m\phi(n)) = x) = p_n(x,y) .\]

Let the continuous time version of $X_n$ be defined by

\[X_n(t) = X_n(m\phi(n)) \text{ if } t \in [m\phi(n), (m+1)\phi(n)].\]

Each $X_n$ is a pure-jump process with jumps occurring only at time values $m\phi(n)$. Let $X_\infty$ be a pure-jump continuous time process which lives on the integers, hence is a random element in $D[0,\infty)$, and assume that $X_\infty$ is non-explosive, i.e., $X_\infty$ has at most finitely many jumps in finite time.
(3.35) **Note:** We'll assume also that the probability under all initial 
(t,x) that $X_\infty$ jumps at some fixed $s > t$ is zero.

Now suppose $X_n \overset{D}{\to} X_\infty$ on $D[0,\infty)$; this type of weak convergence is 
treated in Whitt [ ] and Lindvall [ ]. We want then to relate $F_n$ to 
$F_\infty$ where, as usual

$$F_n(t,x) = \sup_{T \geq t} \int f(T, X_n(T)) dP(t,x), \ n \leq \infty.$$ 

**Assumptions**

(3.36) $f(t,x)$ is increasing in $x$, decreasing in $t$, continuous in $t$.

(3.37) $P(t,x) \left\{ \lim_{s \to \infty} f(s,x_n(s)) = 0 \right\} = 1$, all initial $(t,x)$.

(3.38) There exists non-decreasing $G : [0, \infty)$, with 

$$\Gamma_\infty = \{(t,x) : x \geq G(t) \text{ or } t = \infty\}$$

and we again interpret $T_\infty = \infty$ on paths where $X_\infty(t) < G(t)$, all $t$.

(3.39) For all large $n$, $F_\infty$ is excessive for discrete time $X_n$.

**Theorem (3.3)**

If $X_n, X_\infty$ above satisfy $X_n \overset{D}{\to} X_\infty$ on $D[0,\infty)$, and if (3.35) through 
(3.39) hold, then if $T_n = \text{minimal } m\phi(n)$ where $X_n(m\phi(n)) \geq G(t)$, we 
have

$$F_\infty(t,x) = \lim_{n \to \infty} F_n(t,x) = \lim_{n \to \infty} \int f(T_n, X_n(T_n)) dP(t,x).$$
Proof: We remark, before entering into particulars, that $X_n \overset{D}{\rightarrow} X_\infty$ is intended as weak convergence under all allowable initial $(t,x)$. Also, note (3.35) constitutes an assumption which substitutes for (3.21) of the last theorem in that (3.35) implies that for small $s$, $X_\infty(T_\infty+s) \sim G(T_\infty+s)$ on $\{T_\infty < \infty\}$.

Just as before, we replace $X_n$ be $Y_n$, $n < \infty$ where $X_n \overset{D}{=} Y_n$ and $Y_n \rightarrow Y_\infty$ in the $D[0,\infty)$ metric for each path. Since all paths are integer valued, and since $Y_\infty$ executes at most finitely many jumps in finite time, we see that for a particular path $\omega$, $Y_n(\omega) \rightarrow Y_\infty(\omega)$ if and only if $Y_n$ eventually makes exactly the same jumps as $Y_\infty$, i.e., $Y_n(s^n_m) = Y_\infty(s^\infty_m)$ where $s^n_m$ is the time of the $m$-th jump and further $s^n_m \rightarrow s^\infty_m$. Since $G$ is continuous and since $Y_n(\omega)$ looks like $Y_\infty(\omega)$ except for a slight distortion of the time axis, we have $S_n \rightarrow S_\infty$ on $\{S_\infty < \infty\}$ where $S_n, S_\infty$ are defined as in the last theorem. Also, since $Y_n(S_n) = Y_\infty(S_\infty)$ for large $n$ on $\{S_\infty < \infty\}$, and since $f$ is continuous in $t$, we have, given (3.37), that $f(S_n,Y_n(S_n)) \rightarrow f(S_\infty,Y_\infty(S_\infty))$. Continuing the logic and notation of the previous theorem, noting that $T_n = V_n$, we see that

$$\int f(T_\infty,X_\infty(T_\infty))dP(t,x) \leq \lim_{n \to \infty} \int f(T_n,X_n(T_n))dP(t,x)$$

and that $F_\infty \geq F_n$, so that all conclusions follow. Q.E.D.
REFERENCES


