THE HARMONIC GINI COEFFICIENT AND AFNUENCE INDEXES

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For a society or community, affluence is usually judged by the extent to which individuals enjoy comforts in living and life styles. Whereas poverty is generally quantified in terms of the proportion of the poor people (living below the poverty line) and their income distribution, affluence may likewise be quantified by the proportion of rich people and their income distribution. In this context, a minimal level of (real) income (or wealth) \( w^* \), termed the affluence line, may be set (from various socio-economic considerations) to characterize the rich people (having income not below \( w^* \)), so that an index of affluence may generally be based on \( a^* \), the proportion of rich people, and some other measures of the income distribution of the rich (i.e., the income distribution truncated or censored, from the left, at \( w^* \)). Thus, from the statistical point of view, an affluence picture rests on the upper tail of the income distribution of a society or community (with a cut-off point determined by various socio-economic utility functions), and in this context, it should be kept in mind that generally there may be difficulties in demarcating this cut-off point precisely and also in measuring accurately (or adequately) the real income (or wealth) of (particularly excessively) rich people. Such erroneous mensurations may generally lead to some ambiguous pictures relating to the conventional mean income, the Gini coefficient and other measures of the income distribution of the rich people. These, in turn, affect an index of affluence based on these measures. This difficulty (i.e., the lack of robustness) may largely be avoided by the use of the harmonic mean and the harmonic Gini coefficient of the income distribution of the rich. Thus, the harmonic income gap ratio and the harmonic Gini coefficient play important
roles in the formulation and meaningful interpretations of affluence indexes.

The Gastwirth coefficient (of income inequality) is closely related to the Gini coefficient. In addition, the Gastwirth coefficient, by virtue of a basic invariance property (which may not generally hold for the Gini coefficient), appears to be a robust contender. It is therefore of interest to explore the relationship between these two coefficients and to examine critically the role of the Gastwirth coefficient in the pretext of affluence indexes. Some axiomatic as well as statistical considerations have led to the formulation of various poverty indexes (available in the literature). Because of the intricate relationship of poverty and affluence, counterparts of these indexes for the affluence picture are easy to conceive, although more statistical considerations are needed to enhance robustness of the affluence indexes (in a broad sense). The main objective of the current study is to justify and incorporate the harmonic Gini coefficient and the Gastwirth coefficient in the formulation of some robust affluence indexes.

KEYWORDS. Gastwirth coefficient; harmonic mean; income gap ratio; Lorenz curve; poverty; robustness; utility theory.

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1. INTRODUCTION

It will be convenient for us to present, side by side, some of the poverty indexes, and to incorporate them in the motivation and formulation of parallel indexes of affluence.

For a society or community, let $F = \{F(y), y \in R^+ = [0, \infty]\}$ be the income distribution, so that for a set poverty line $\omega(>0)$, $a = F(\omega)$ represents the proportion of poor people. Similarly, for a set affluence line $\omega^*(0 < \omega < \omega^* < \infty)$,

$$a^* = F(\omega^*) = 1 - F(\omega^*) = \text{proportion of the rich.} \quad (1.1)$$

The average income of the poor is $\mu_a = a^{-1}\int_0^\omega ydF(y)$, so that the income gap ratio ($\gamma$) of the poor is equal to

$$\gamma = 1 - \omega^{-1}\mu_a = 1 - \omega^{-1}a^{-1}\int_0^\omega ydF(y). \quad (1.2)$$

Also, let $G_\alpha$ be the classical Gini coefficient of the income distribution of the poor (i.e., the Gini coefficient of the truncated distribution $F_\alpha$; $F_\alpha(y) = a^{-1}F(y)$ for $y \leq \omega$ and $F_\alpha(y) = 1$, $y > \omega$). Then, typically, an index of poverty ($\pi$) is based on the triplet $(a, \beta, G_\alpha)$ i.e., $\pi = \pi(a, \beta, G_\alpha)$. The following two forms are due to Sen (1976):

$$\pi_\alpha = a\beta = a - \omega^{-1}\int_0^\omega ydF(y), \quad (1.3)$$

$$\pi_\alpha^* = a\beta + (1 - \beta)G_\alpha; \quad (1.4)$$

a robust version of $\pi_\alpha^*$, considered by Sen (1986), is the following:

$$\pi_\alpha^* = a(\beta^{1-G_\alpha}). \quad (1.5)$$

There are other poverty indexes too, and a detailed study of the relative merits and demerits of some of these indexes is due to Sen (1986).

For the affluent people, the income is bounded from below by $\omega^*$, so that the average income $\mu_{\alpha^*} = (a^*)^{-1}\int_{\omega^*}^\infty ydF(y)$ is $\omega^*$. Hence, the income gap ratio ($\beta^*$) of the rich people needs to be defined in a manner different from (1.2). Also, parallel to $G_\alpha$, let $G_\alpha^*$ be a suitable measure of the income inequality of the rich people. Some specific choice of $\beta^*$ and $G_\alpha^*$ will be discussed later on. Then, based on the triplet $(a^*, \beta^*, G_\alpha^*)$, we may formulate
an index of affluence $\gamma = \gamma(\alpha^*, \beta^*, G_{\alpha^*}^*)$, in a manner similar to the case of poverty indexes. Specifically, parallel to (1.3), (1.4) and (1.5), we propose the following three indexes:

$$
Y_A = y_A(\alpha^*, \beta^*) = \alpha^* \beta^* \\
Y_S = \alpha^* \{\beta^* + (1 - \beta^*) G_{\alpha^*}^*\}, \\
Y^* = \alpha^* \{(\beta^*)^{1-G_{\alpha^*}^*}\}.
$$

Once the measures $\beta^*$ and $G_{\alpha^*}^*$ are justified on suitable grounds, we may proceed as in Sen (1986) and study the relative merits and demerits of these indexes. The crux of the problem is therefore to scrutinize the rationality of the choice of suitable $\beta^*$ and $G_{\alpha^*}^*$.

As has been remarked earlier that $\mu_{\alpha^*} > \omega^*$, and hence, one possibility of defining the income gap ratio $\beta^*$ is to set

$$
\beta^*_l = 1 - \omega^*/\mu_{\alpha^*} = 1 - \omega^* \alpha^* (\int_{\omega^*}^{\infty} y dF(y))^{-1}.
$$

Similarly, for the coefficient $G_{\alpha^*}^*$, one may as well choose the Gini coefficient of the income distribution among the rich. If we define the truncated (from left) distribution $F_{\alpha^*}^*$ by

$$
F_{\alpha^*}^*(y) = \begin{cases} 
0, & y < \omega^*, \\
(\alpha^*)^{-1} [F(y) - F(\omega^*)], & y \geq \omega^*,
\end{cases}
$$

then, following the treatment of Sen (1986), we may set

$$
G_{\alpha^*}^* = \{E[Y_1^* - Y_2^*]/\{E(Y_1^* + Y_2^*)\},
$$

where $Y_1^*$ and $Y_2^*$ are two independent random variables, each one having the distribution $F_{\alpha^*}^*$, defined by (1.10). Note that both (1.9) and (1.11) involve the first moment $EY_1^*$, and, in addition, (1.11) involves a more involved moment $E|Y_1^* - Y_2^*|$, for the left truncated distribution $F_{\alpha^*}^*$. For income distributions with some degree of uncertainty for reliable mensuration of 'high' incomes, both (1.9) and (1.11) are quite vulnerable to gross errors and outliers. Therefore, it may be wiser to incorporate some alternative measures (of the income gap ratio
and the income inequality among the rich) which are less susceptible to gross
errors and outliers. From the (economic) utility theory point of view also,
(1.9) and (1.11) are not the most desirable choices. Thus, the primary objective
of the current study is to focus on this choice of $\beta^*$ and $G_{a^*}$, and towards this,
the harmonic mean and the harmonic Gini coefficient provide robust alternative
choices. These are considered in Section 2. Closely related to the Gini coef-
ficient is another measure of income inequality due to Gastwirth (1975). It
appears that the Gastwirth coefficient possesses an important invariance property
(not generally shared by the Gini coefficient). The relationship between these
two coefficients is explored in Section 3. Some general remarks are made in
the concluding Section.

2. THE HARMONIC GINI COEFFICIENT

We may observe that in (1.2), we have taken the ratio $\omega^{-1} \mu_a$, while in (1.9),
we have $\omega^*/\mu_{a^*}$. This suggests that instead of the reciprocal of the arithmetic
mean $\mu_{a^*}$, it may also be plausible to work with the harmonic mean $\mu_{a^*}^H$, where

$$
\mu_{a^*}^H = \{ (a^*)^{-1} \int_{0}^{\infty} y^{-1} dF(y) \}^{-1} = a^* / \int_{0}^{\infty} y^{-1} dF(y).
$$

(2.1)

As such, we may consider a second measure of the income gap ratio among the
rich as

$$
\beta_2^* = 1 - \omega^*/\mu_{a^*}^H = 1 - (\omega^*/a^*) \int_{0}^{\infty} y^{-1} dF(y).
$$

(2.2)

$\beta_2^*$ has also a second interpretation. By nature, poverty and affluence are of
opposite order. An excessive income in the context of affluence seems to be
the counterpart of a very low income in the context of poverty. Hence, we may
as well work with the reciprocal of the incomes of the rich people, and, by
drawing analogy with the poverty indexes, we may define the affluence indexes
on them. Note that $Y^*$ [in (1.11)] stands for the income variable of the rich
(truncated from below by $\omega^*$), so that we may work with $Z^* = (Y^*)^{-1}$, and then
$Z^*$ is bounded from above by $1/\omega^*$. On these $Z^*$, if we adopt the classical
definition in (1.2), the corresponding income gap ratio is given by

\[ 1 - (1/\omega) E(Z) = 1 - \omega E(Y^{-1}) \]

\[ = 1 - (\omega/\alpha) \int_{\omega}^{\infty} y^{-1} dF(y) = \beta^*_2. \quad (2.3) \]

Next, we note that by definition (for nonnegative rent's),

\[ E(Y^{-1}) = (\alpha)^{-1} \int_{\omega}^{\infty} y^{-1} dF(y) \]

\[ \geq (E(Y^*))^{-1} = \alpha/\int_{\omega}^{\infty} y dF(y), \quad (2.4) \]

so that

\[ \omega/\nu(H^*) = (\omega/\alpha) \int_{\omega}^{\infty} y^{-1} dF(y) \geq \omega \alpha (\int_{\omega}^{\infty} y dF(y))^{-1}, \quad (2.5) \]

which, in turn, implies that

\[ \beta^*_2 = 1 - \omega/\nu(H^*) \leq 1 - \omega \alpha (\int_{\omega}^{\infty} y dF(y))^{-1} = \beta^*_1, \quad (2.6) \]

where the equality sign holds when the income distribution \( F^*_\alpha \) is degenerate.

As such, in (1.6), (1.7) or (1.8), the use of \( \beta^*_2 \) instead of \( \beta^*_1 \) (for \( \beta^*_\alpha \)) will generally lead to a smaller numerical value for the affluence index. This feature is somewhat comparable to the relative picture of \( \pi^*_S \) and \( \pi^* \), studied in detail in Sen (1986).

Let us now consider the income inequality measure \( G^*_\alpha \). A natural candidate for this measure is the Gini coefficient of the income distribution truncated from below by \( \omega^* \) and is defined by (1.10) - (1.11). We have already remarked that (1.11) is quite vulnerable to "gross errors" and outliers, and in the actual recording of 'high' incomes, we generally have gross errors as well as outliers. One way of defusing this susceptibility of the Gini coefficient is to curb the excessive influence of 'high incomes' through suitable transformation on the income variable. Since for the affluence picture, we deal with the upper tail of the income distribution (which is generally very skewed), \( F^*_\alpha \) is rarely expected to be of a symmetric form, so that the usual (Huber-type) bounded transformations used in the robust estimation of location/regression parameters may not be generally meaningful or appropriate here. Drawing analogy with the case of the income gap ratio \( \beta^*_2 \), it seems quite plausible to work with
the reciprocal income variable (i.e., \( Z^* = (Y^*)^{-1} \)) and consider the usual Gini coefficient on these transformed variables. Thus, we take [parallel to (1.11)]

\[
G^H_{\alpha^*} = \frac{\{EZ_1^* - Z_2^*\}}{\{EZ_1^* + EZ_2^*\}}
= \frac{\{E(Y_1^* - Y_2^*)/(Y_1^* Y_2^*)\}}{\{E[(Y_1^* + Y_2^*)/(Y_1^* Y_2^*)]\}},
\]

(2.7)

where \( Y_1^* \) and \( Y_2^* \) are independent r.v.'s, each having the distribution \( F_{\alpha^*} \). We term \( G^H_{\alpha^*} \) as the harmonic Gini coefficient for the rich.

Let us provide a second interpretation to \( G^H_{\alpha^*} \). From the utility theory point of view, the marginal value of money (income) is decreasing for any person. Thus, the difference \( |Y_1 - Y_2| \) may not be equally significant at all levels of \( Y_1 \) and \( Y_2 \). Rather, \( |Y_1 - Y_2| \) has a real value which goes down as \( Y_1 \) (and \( Y_2 \)) increase. Hence, it may be quite natural to introduce a utility function \( u(t_1, t_2) \), \( t_1 \geq 0 \), \( t_2 \geq 0 \), such that \( u(t_1, t_2) \) is non-negative and is non-increasing in each of \( t_1 \) and \( t_2 \), and then by analogy with (1.11), define a utility-oriented Gini coefficient as

\[
G^u_{\alpha^*} = \frac{\{E[|Y_1^* - Y_2^*|u(Y_1^*, Y_2^*)]\}}{\{E[(Y_1^* + Y_2^*)u(Y_1^*, Y_2^*)]\}}
\]

(2.8)

[The case of \( u(t_1, t_2) = 1 \) relates to (1.11).] If, in particular, we consider a harmonic utility function \( u(t_1, t_2) = t_1^{-1} t_2^{-1} \), by (2.8) and (2.7), we conclude that \( G^u_{\alpha^*} \) reduces to \( G^H_{\alpha^*} \).

For the study of affluence (in a quantitative manner), we advocate the use of the harmonic coefficients in (2.2) and (2.7). Thus, for (1.6), (1.7) or (1.8), we prescribe the choice of \( \beta^*_2 \) and \( G^H_{\alpha^*} \) for \( \beta^* \) and \( G^*_\alpha \), respectively. The main advantage of using the harmonic coefficients is that an alteration in a 'high income' has generally profound influence on (1.9) and (1.11), while (2.2) and (2.7) are relatively more insensitive to such gross errors and outliers.

We have already noticed [in (2.5) - (2.6)] that the harmonic income gap ratio is more conservative than (1.9). Comparing (1.11) and (2.7), we notice that
\[
G_{\alpha^*}/G_{\alpha} \text{ is } \geq 1 \text{ according as }
\]
\[
E(Y_1^* + Y_2^*)E[|Y_1^* - Y_2^*|/Y_1^*Y_2^*] \geq \frac{1}{2} E|Y_1^* - Y_2^*|E[(Y_1^* + Y_2^*)/Y_1^*Y_2^*].
\]

Verification of the inequality sign (either way) in (2.9) is more involved, and it rests on the behavior of the four expectations involved in (2.9). As in Sen (1986), we may rewrite
\[
G_{\alpha^*}^H = 1 - (EY_1^*E)E(Y_1^*Y_2^*),
\]
\[
G_{\alpha^*}^H = 1 - (EY_1^*E)E(Y_1^*Y_2^*),
\]
so that
\[
1 - G_{\alpha^*}^H = \frac{E(Y_1^*-1Y_2^*)E(Y_1^*)}{1 - G_{\alpha^*}^H} = \frac{E((Y_1^*VY_2^*)^{-1})E(Y_1^*)}{E(Y_1^*Y_2^*)E(Y_1^*)}
\]
\[
= \frac{E(Y_1^*VY_2^*)E((Y_1^*VY_2^*)^{-1})E(Y_1^*)}{E(Y_1^*Y_2^*)E(Y_1^*)}
\]
\[
(2.10)
\]
\[
(2.11)
\]
\[
(2.12)
\]
Now, for any nonnegative \(r \cdot v \cdot Z\), \(E(Z)E(Z^{-1}) \geq 1\). Also, \((Y_1^* \vee Y_2^*)\) and \((Y_1^* \wedge Y_2^*)\) are positively correlated, while \((Y_1^* \vee Y_2^*)\) and \((Y_1^* \wedge Y_2^*)\) are positively correlated, while \((Y_1^* \vee Y_2^*)\) and \((Y_1^* \wedge Y_2^*)\) are positively correlated, while (2.12) is bounded from below by
\[
E(Y_1^*)/(\{E(Y_1^*)\}^2E(Y_1^*)^{-1}) = [E(Y_1^*)E(Y_1^*)^{-1}]^{-1} = \gamma, \text{ say,}
\]
where, of course, \(0 < \gamma \leq 1\). Similarly,
\[
(1 - G_{\alpha^*}^H)/(1 - G_{\alpha^*}^H) \geq [E(Y_1^*)E(Y_1^*)^{-1}]^{-1} = \gamma.
\]
From (2.12) through (2.14), we obtain that
\[
\gamma^{-1} \geq (1 - G_{\alpha^*}^H)/(1 - G_{\alpha^*}^H) \geq \gamma,
\]
which leads us to
\[
1 - \gamma^{-1}(1 - G_{\alpha^*}^H) \leq G_{\alpha^*}^H \leq 1 - \gamma(1 - G_{\alpha^*}^H).
\]
\[
(2.15)
\]
\[
(2.16)
\]
Note that \(\gamma = (\text{Harmonic mean of } Y^*)/(\text{Arithmetic mean of } Y^*)\), and this ratio
(0 < \gamma \leq 1) provides the simple inequality in (2.16). In particular, (2.16) shows that whenever \gamma is "close to" 1, \( G^*_a \) and \( G^H_a \) are also close to each other.

If \( F^*_a \) has the form of the classical Pareto distribution (i.e., \( F^*_a (y) = 1 - (y/\omega^*)^{-\upsilon^*}, y \geq \omega^*, \) where \( \upsilon^* (>0) \) stands for the income inequality parameter), then it is easy to verify that for \( \upsilon^* > 1, G^*_a = (2\upsilon^* - 1)^{-1} \) while \( G^H_a = (2\upsilon^* + 1)^{-1} \), so that \( G^H_a < G^*_a \), \( \upsilon^* > 1 \); for \( \upsilon^* \leq 1, G^*_a \) does not exist, while \( G^H_a \) does. In general, for the affluence picture, the upper tail of the income distribution is highly skewed, while the reciprocal variable has a less skewed distribution (on a compact interval \([0, \omega^*-1]\)), and, we would generally have \( G^H_a \leq G^*_a \). However, such an inequality may not be generally true for all \( F^*_a \).

To illustrate this point, consider a special case where \( F^*_a \) has only two mass points \( \omega_1 \) and \( \omega_2 \) (where \( \omega_2 > \omega_1 > \omega^* \)) with respective probability masses \( p \) and \( q (= 1 - p) \). Then, by (2.10) and (2.11),

\[
G^*_a = 1 - \frac{(\omega_2 - \omega_1)pq}{\omega_1 p + \omega_2 q},
\]

\[
G^H_a = 1 - \frac{(\omega_2 - \omega_1)pq}{\omega_1 q + \omega_2 p}.
\]

Then

\[
\frac{G^H_a}{G^*_a} = \frac{(\omega_1 p + \omega_2 q)/(\omega_1 q + \omega_2 p)}{[\omega_1 + (\omega_2 - \omega_1)p]/[\omega_1 + (\omega_2 - \omega_1)p]},
\]

and this can always be made >, = or < 1, by varying \( p \) (and \( q \)). In particular, for \( p > \frac{1}{2}, G^H_a < G^*_a, p = \frac{1}{2}, G^H_a = G^*_a, \) and for \( p < \frac{1}{2}, G^H_a > G^*_a \). This also shows that in (2.9), a universal inequality does not hold.

3. THE GASTWIRTH COEFFICIENT

As a measure of income inequality, Gastwirth (1975) has considered a coefficient, closely related to the Gini coefficient. With the notations in (1.10) - (1.11), the Gastwirth coefficient (for the income distribution \( F^*_a \)) is defined by

\[
G^0_a = E\{|Y_1^* - Y_2^*|/(Y_1^* + Y_2^*)\}. \quad (3.1)
\]
Unlike the case of the Gini coefficient, the Gastwirth coefficient may not be directly obtained from the Lorenz curve for the associated income distribution. Nevertheless, it has some nice properties which may not always hold for the Gini coefficient. A natural estimator of $G^*_a$ in (1.11) is obtained by replacing the expectations by their sample counterparts. This estimator is thus a ratio of two U-statistics [viz., Hoeffding (1948)]. However, being a ratio-estimator, it is generally not unbiased for $G^*_a$. On the other hand, for $G^O_{\alpha^*}$, the sample counterpart is a single U-statistic and is necessarily unbiased for $G^O_{\alpha^*}$. Secondly, we have noticed in Section 2 that generally $G^*_a$ and $G^H_{\alpha^*}$ are not the same. In this respect, we may note that the harmonic Gastwirth coefficient $G^H_{\alpha^*}$ is given by

$$G^H_{\alpha^*} = E\{Y_1^{-1} - Y_2^{-1} / (Y_1^{-1} + Y_2^{-1})\}. \quad (3.2)$$

But, note that for nonnegative $a, b$,

$$|a^{-1} - b^{-1}|/(a^{-1} + b^{-1}) = |a-b|/ab \cdot a+b = |a-b|/a+b, \quad (3.3)$$

so that by (3.1), (3.2) and (3.3), we have

$$G^H_{\alpha^*} = G^O_{\alpha^*}, \quad \forall \alpha^* \in \mathbb{R}^+. \quad (3.4)$$

Thus, the Gastwirth coefficient remains invariant under the transformation:

$$Y^* \longrightarrow \ Y^* - 1. \quad \text{This invariance property of } G^O_{\alpha^*} \text{ makes it even more attractive in the context of affluence indexes.}$$

Let us now examine the relationship between $G^*_a$ and $G^O_{\alpha^*}$. For this, we define the random variables $T = Y_1^* + Y_2^*$ and $U = Y_1^*/T^*$. Then by (3.1) and (3.5), we have

$$G^O_{\alpha^*} = \frac{E|2U - 1|}{2}, \quad (3.5)$$

while, by (1.11)

$$G^*_a = \frac{E(T|2U - 1|)}{E(T)}. \quad (3.6)$$

Note that two random variables (say, $X$ and $Y$) are said to be positively (or negatively) associated according as $\text{Cov}(X,Y)$ is positive (or negative), i.e.,
according as $E(XY)$ is $\geq$ (or $\leq$) $E(X)E(Y)$. Since $T$ is nonnegative, while $U \in [0,1]$, by (3.5) and (3.6), we arrive at the following:

$$G_{\alpha^*} \geq \frac{\alpha^*}{\alpha^*} G_{\alpha^*}$$ according as $T$ and $|2U - 1|$ are positively or negatively associated. \hfill (3.7)

In a similar manner, let $T^* = (Y_1^*)^{-1} + (Y_2^*)^{-1}$ and $U^* = (Y^*)^{-1}/T^*$. Then by (3.2), (3.4) and (2.7), we have

$$G_{\alpha^*}^H = \frac{E(T^*|2U^* - 1|)}{E(T^*)}, \quad G_{\alpha^*}^{OH} = E|2U^* - 1|.$$ \hfill (3.8)

Thus, parallel to (3.7), we have the following:

$$G_{\alpha^*}^H \geq \frac{\alpha^*}{\alpha^*} G_{\alpha^*}^{OH} = G_{\alpha^*}^{OH}$$ according as $T^*$ and $|2U^* - 1|$ are positively or negatively associated. \hfill (3.9)

It may be noted that

$$U^* = (Y^*)^{-1}/[(Y_1^* + Y_2^*)/(Y_1^*Y_2^*)] = Y_2^*/(Y_1^* + Y_2^*) = 1 - U,$$ \hfill (3.10)

so that $|2U^* - 1| = |2(1 - U) - 1| = |2U - 1|$. Consequently, in (3.9), $|2U^* - 1|$ may also be replaced by $|2U - 1|$.

Let us illustrate (3.7) and (3.9) by some examples. First, consider the case where $F_{\alpha^*}$ has the gamma distribution. In this case, $U$ has the beta distribution, and $U$ is independent of $T$ (having also a gamma distribution), so that in (3.7), we have $G_{\alpha^*} = G_{\alpha^*}^{OH}$. As a second example, consider the classical Pareto distribution for $F_{\alpha^*}$. Again $U^*$ has the beta distribution, independent of $T^*$, and hence, in (3.9), we have $G_{\alpha^*}^H = G_{\alpha^*}^{OH} = G_{\alpha^*}^{OH}$. For other forms of $F_{\alpha^*}$, (3.7) or (3.9) may be used to study the relative order of the three coefficients $G_{\alpha^*}$, $G_{\alpha^*}^H$ and $G_{\alpha^*}^{OH}$. However, a completely nonparametric characterization of their relative ordering is not known.

4. SOME GENERAL REMARKS

Let us look back at the three affluence indexes in (1.6), (1.7) and (1.8). By virtue of (2.6), it seems more reasonable to use $\beta^*$ for $\beta^*$. Also, proceeding as in Sen (1986), we obtain that

$$Y_A \leq Y^* \leq Y_S \leq (\leq \alpha^*), \text{ for all } F_{\alpha^*}^*.$$ \hfill (4.1)
Following arguments similar to those in Sen (1986), we may then advocate the use of $\gamma^*$ in (1.8) as a suitable affluence index. In this context, when we propose the use of $\beta_2^*$ for $\beta^*$, the question remains open as to the suitable choice of $G_{12}^*$. Based on the results in Sections 2 and 3, we prefer to use $G_{12}^H$ or $G_{12}^O$ over $G_{12}^*$ (on the ground of robustness); of these two, $G_{12}^O$ may again be preferred on the ground of its invariance (under harmonic transformation).

We illustrate this relative picture for the special case where $F_{12}^*$ is a Pareto distribution, i.e., $F_{12}^*(x) = 1 - (x/w^*)^{-v^*}$, $x > w^*$, where $v^* (> 0)$ is the income inequality parameter. Note that here

$$
\beta_2^* = 1 - w^*(\int_{0}^{\infty} y^{v^*} dy)^{-1} = (1 + v^*)^{-1},
$$

$$
\beta_2^* = 1 - w^*(\int_{w^*}^{\infty} y^{v^*} dy)^{-1} = (v^*)^{-1},
$$

$$
G_{12}^* = (2v^* - 1)^{-1} \text{ and } G_{12}^H = G_{12}^O = (2v^* + 1)^{-1}.
$$

Thus, based on $\beta_2^*$, $G_{12}^H$ (or $G_{12}^O$), $\gamma^*$ in (1.8) reduces to

$$
\gamma^O = \alpha^*((\beta_2^*)^{1-G_{12}^O}) = \alpha^*(1 + v^*)^{-2v^*/(1 + 2v^*)},
$$

On the other hand, for $G_{12}^*$, (with $v^* > 1$), we have for $\gamma^*$,

$$
\gamma^* = \alpha^*((\beta_1^*)^{1-G_{12}^*}) = \alpha^*(1 + v^*)^{-2(v^*-1)/(2v^*-1)}.
$$

Thus, $\gamma^O < \gamma^*$ for every $v^* > 1$. Similarly, for $\gamma^S$, we have

$$
\gamma^O_S = \alpha^*(\beta_2^* + (1 - \beta_2^*)G_{12}^O) = \alpha^*(3v^* - 2)/(v^*(2v^* - 1)),
$$

$$
\gamma^O_S = \alpha^*(\beta_1^* + (1 - \beta_1^*)G_{12}^O) = \alpha^*(3v^* + 1)/(v^* + 1)(2v^* + 1),
$$

so that $\gamma^O_S < \gamma^O_S$ for every $v^* > 1$. Also, note that for $v^* > 0$,

$$
\frac{\gamma^O_{\gamma^S}}{\gamma^O_S} = (3v^* + 1)^{-1}(2v^* + 1)/(1 + v^*)^{1/(2v^* + 1)},
$$

where as $v^* \to 0$, (4.6) converges to 1, while as $v^*$ increases, (4.6) converges to its lower asymptote 2/3. Thus, depending on the value of the parameter $v^*$, the two indexes $\gamma^O$ and $\gamma^O_S$ may differ considerably. Higher discrepancy may be
observed for $\gamma^* / \gamma_S$. Comparing (4.2) and (4.4), we may conclude that as $2v^*/(1 + 2v^*) < 1$, $v^* < \infty$, $\gamma^*_o$ behaves more steadily for large values of $v^*$ (than $\gamma^*_S$). This may provide us with some intuitive reason for prescribing $\gamma^*_o$ as an appropriate index of affluence.

REFERENCES


