TEST 1 Solutions

1. The joint distribution of the number of college football games and the number of pro football games broadcast on a random fall weekend is

<table>
<thead>
<tr>
<th></th>
<th>College</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>.00</td>
<td>.20</td>
</tr>
<tr>
<td>Pro</td>
<td>2</td>
<td>.15</td>
</tr>
<tr>
<td>3</td>
<td>.05</td>
<td>.15</td>
</tr>
</tbody>
</table>

(a) What is the probability that there are more pro games than college games?

**Solution:**
Write \( X \) for the number of pro games and \( Y \) for the number of college games.

\[
P(X > Y) = P(X = 2 \cap Y = 1) + P(X = 3 \cap Y = 1) + P(X = 3 \cap Y = 2)
\]
\[
= .15 + .05 + .15
\]
\[
= .35
\]

(b) What is the probability that the total number of games is at least 5?

**Solution:**

\[
P(X + Y \geq 5) = P(X = 2 \cap Y = 3) + P(X = 3 \cap Y = 2) + P(X = 3 \cap Y = 3)
\]
\[
= .05 + .15 + .15
\]
\[
= .35
\]
(c) What is the expected value of the total number of games?

**Solution:**
The pmf for $Z = X + Y$ is:

<table>
<thead>
<tr>
<th>$z$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(z)$</td>
<td>.35</td>
<td>.30</td>
<td>.20</td>
<td>.15</td>
<td></td>
</tr>
</tbody>
</table>

So

$$E(X + Y) = 2 \times 0 + 3 \times .35 + 4 \times .30 + 5 \times .20 + 6 \times .15$$
$$= 0.00 + 1.05 + 1.20 + 1.00 + 0.90$$
$$= 4.15$$

2. You are planning a day-trip to Charlotte for a job interview. The chance of a delay on the morning trip is 15%, and the chance of a delay on the return trip that evening is 20%. Assume that these events are independent.

(a) If you are not held up in the morning, what is the chance that you will not be held up in the evening? What principle do you use to calculate the probability?

**Solution:**
Write $A$ for the event that you are held up in the morning, and $B$ for the event that you are held up in the evening. The probability is conditional on “If you are not held up in the morning”, that is $A'$:

$$P(B'|A') = P(B') \quad \text{by independence}$$
$$= 1 - P(B)$$
$$= .8$$

Alternatively,

$$P(B'|A') = \frac{P(B' \cap A')}{P(A')}$$
$$= \frac{P(B')P(A')}{P(A')} \quad \text{by independence}$$
$$= P(B') = 1 - P(B) = .8$$

(b) What is the chance of making the trip with no delays?

**Solution:**

$$P(A' \cap B') = P(A')P(B') = .85 \times .8 = .68$$
(c) If you do have a delay, what is the chance that both legs are delayed?

**Solution:**
Again, the probability is conditional on “If you do have a delay”, that is $A \cup B$:

$$P(A \cap B | A \cup B) = \frac{P[(A \cap B) \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A \cap B)}{P(A \cup B)}$$

because $(A \cap B) \subseteq (A \cup B)$

$$= \frac{P(A)P(B)}{P(A) + P(B) - P(A)P(B)}$$

$$= \frac{.03}{.32}$$

$$= .09375$$

3. Displays on a particular notebook computer carry a one-year warranty, and on average 5% are repaired under warranty. A lab purchases such notebooks for 6 research assistants.

(a) What is the probability that no more than 1 will be repaired under warranty?

**Solution:**
Write $X$ for the number needing repair. On the assumption that events of repair are independent, $X \sim \text{Bin}(6, .05)$.

$$P(X = 0) = .95^6 = .7351$$

$$P(X = 1) = 6 \times .05 \times .95^5$$

$$= \frac{6 \times .05}{.95} P(X = 0) = .2321$$

so

$$P(X \leq 1) = .7351 + .2321 = .9672$$

(b) If each notebook returned for repair costs $20.00 in postage and packing, what are the expected value and standard deviation of the cost to the lab?

**Solution:**

$$E(20X) = 20E(X) = 20np = 6,$$
and

$$\sigma_{20X} = 20|\sigma_X = 20 \sqrt{np(1-p)} = 10.68,$$

so the mean and standard deviation are $6.00$ and $10.68$, respectively.

4. The amount of rainfall measured at a rain gauge during the month of October is normally distributed with mean 3 inches and standard deviation 0.8 inches. What is the probability

(a) Of observing more than 4 inches?

Solution:
Write $X$ for the amount of rainfall, and

$$Z = \frac{X - 3}{0.8},$$

so that $Z \sim N(0, 1)$. Then

$$P(X > 4) = P(Z > 1.25)$$
$$= 1 - \Phi(1.25)$$
$$= 1 - .8944$$
$$=.1056$$

(b) Of observing between 3.5 and 4 inches?

Solution:

$$P(3.5 < X < 4) = P(0.625 < Z < 1.25)$$
$$= \Phi(1.25) - \Phi(0.625)$$
$$\approx \Phi(1.25) - \Phi(0.63) \quad \text{(rounding up)}$$
$$= .8944 - .7357$$
$$=.1587$$

You could also round down: $\Phi(0.625) \approx \Phi(0.62) = .7324$; or interpolate:

$$\Phi(0.625) \approx \frac{\Phi(0.62) + \Phi(0.63)}{2} = .7341$$

(c) That the rainfall is within 1 inch of the mean?

Solution:

$$P(|X - 3| \leq 1) = P(|Z| \leq 1.25)$$
$$= \Phi(1.25) - \Phi(-1.25)$$
$$= \Phi(1.25) - [1 - \Phi(1.25)]$$
$$= 2 \times \Phi(1.25) - 1$$
$$= .7888$$
5. A team of students compared two brands of battery by measuring their operational life in a bicycle headlight. When batteries of brand X are used, the lifetime is normally distributed with mean 15 hours and standard deviation 2 hours; when batteries of brand Y are used, the lifetime is normally distributed with mean 18 hours and standard deviation 1.5 hours. One battery of each brand was tested. You can assume that lifetimes in different tests are independent.

(a) If a single headlight is used and the brands are tested consecutively (first brand X and then, when the first battery is exhausted, brand Y), what is the probability that the whole experiment will last 40 hours or longer?

Solution:
Write $X$ and $Y$ for the lifetimes of the brand X battery and the brand Y battery, respectively. Then
\[ X + Y \sim N(15 + 18, 2.0^2 + 1.5^2) = N(33, 2.5^2), \]
so
\[ Z = \frac{X + Y - 33}{2.5} \sim N(0, 1), \]
and
\[ P(X + Y \geq 40) = P(Z \geq 2.8) \]
\[ = 1 - \Phi(2.8) \]
\[ = .0026 \]

(b) If two identical headlights are used and the tests are started at the same time, what is the chance that the battery of brand X will outlast the battery of brand Y?

Solution:
We want
\[ P(X > Y) = P(X - Y > 0). \]
Now
\[ X - Y \sim N(15 - 18, 2.0^2 + 1.5^2) = N(-3, 2.5^2), \]
so
\[ Z = \frac{X - Y - (-3)}{2.5} \sim N(0, 1), \]
and
\[ P(X - Y > 0) = P(Z > 1.2) \]
\[ = 1 - \Phi(1.2) \]
\[ = .1151 \]