Exponentially Weighted Moving Average Chart

Given a sequence of observations \( x_1, x_2, \ldots, x_n \), the **exponentially weighted moving average** (EWMA) is defined recursively by

\[
z_i = \lambda x_i + (1 - \lambda)z_{i-1}, \quad i = 1, 2, \ldots, n
\]

where \( 0 < \lambda \leq 1 \) is a constant, and the starting value is the process target: \( z_0 = \mu_0 \).

Successive substitution shows that

\[
z_i = \sum_{j=0}^{i-1} \lambda(1 - \lambda)^j x_{i-j} + (1 - \lambda)^i z_0,
\]

a weighted average of \( x_i, x_{i-1}, \ldots, x_1, z_0 \).
So if $E(X_i) = \mu_0$, then $E(Z_i) = \mu_0$, and if the observations $X_i$ are uncorrelated random variables with variance $\sigma^2$, then

$$\text{Var}(Z_i) = \sigma^2 \left( \frac{\lambda}{2 - \lambda} \right) \left[ 1 - (1 - \lambda)^{2i} \right].$$

We can set up an $L$-sigma control chart for $z_i$:

Upper Control Limit (UCL) = $\mu_0 + L\sigma \sqrt{\left( \frac{\lambda}{2 - \lambda} \right) \left[ 1 - (1 - \lambda)^{2i} \right]}$

Center line = $\mu_0$

Lower Control Limit (LCL) = $\mu_0 - L\sigma \sqrt{\left( \frac{\lambda}{2 - \lambda} \right) \left[ 1 - (1 - \lambda)^{2i} \right]}$
In R:
Use the simulated data used as an example of the cusum chart; recall that $\mu_0 = 10$ and $\sigma = 1$; use $\lambda = 0.1$ and $L = 2.7$:

```r
library(qcc)
summary(ewma(x, center = 10, std.dev = 1,
    lambda = 0.1, nsigmas = 2.7))
```

Notes

- The control limits change with $i$, but approach asymptotes.
- The same two observations fall outside the control limits as in the Cusum control chart.
Design

Like the cusum chart, the EWMA chart’s properties are difficult to calculate, so the focus is on the ARL. The performance of the chart is determined by $L$ and $\lambda$.

Suppose we want an EWMA chart with a specified $ARL_0$, say 370, and a low $ARL_1$ for a one-sigma shift.

For any choice of $\lambda$, the function `xewma.crit(l, L0)` in the `spc` package computes the $L$ that gives the desired $ARL_0$, $L0$, for a given $\lambda$, $l$. 
Optimal design

```r
o <- optimize(function(lambda)
    xewma.arl(l = lambda,
        c = xewma.crit(lambda, 370, sided = "two"),
        mu = 1, sided = "two"),
        interval = c(0, 1))

print(o)
xewma.crit(o$minimum, 370, sided = "two")
```

The optimal $\lambda = 0.14$ and control limits $L = 2.79$, with $ARL_1 = 9.58$.

The performance is very similar to, but slightly better than, the optimal cusum chart (with no head start), for which the $ARL_1$ was 9.92.
Moving Average Control Chart

The EWMA chart is easy to set up recursively, since the current value of $z$ depends on only the current $x$ and the previous $z$.

However, because it is a (exponentially) weighted average of all past observations, it is impacted, at least a little, by the entire past.

An alternative, with finite memory, is the moving average (MA) chart with span $w$, based on the unweighted average

$$m_i = \frac{x_i + x_{i-1} + \cdots + x_{i-w+1}}{w} \quad i = w, w+1, \ldots.$$
For $i < w$, we define $m_i$ to be the average of $x_1, x_2, \ldots, x_i$.

If $E(X_i) = \mu_0$ and the $X_s$ are uncorrelated with variance $\sigma^2$, then

$$E(M_i) = \mu_0,$$

and

$$\text{Var}(M_i) = \frac{\sigma^2}{\min(w, i)}.$$
So we can construct a control chart for \( m_i \) with

\[
UCL = \mu_0 + L\sigma \sqrt{\frac{1}{\min(w, i)}}
\]

Center line = \( \mu_0 \)

\[
LCL = \mu_0 - L\sigma \sqrt{\frac{1}{\min(w, i)}}
\]

Like the EWMA chart, the limits change with \( i \), but are \textit{constant} after \( i \geq w \), instead of \textit{asymptoting} to a constant.

See Figure 9.8; the qcc package does not provide MA control charts.