Vector-Valued Series–Notation

- Studies of time series data often involve $p > 1$ series.

- E.g. Southern Oscillation Index and recruitment in a fish population ($p = 2$).

- Treated as a $p \times 1$ column vector:

$$
x_t = \begin{pmatrix}
    x_{t,1} \\
    x_{t,2} \\
    \vdots \\
    x_{t,p}
\end{pmatrix}
$$
Mean Vector

• Assume jointly weakly stationary.

• mean vector:

\[ \mathbf{\mu} = \mathbf{E}(\mathbf{x}_t) = \begin{pmatrix} E(x_{t,1}) \\ E(x_{t,2}) \\ \vdots \\ E(x_{t,p}) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{pmatrix} \]
Autocovariance Matrix

- Autocovariance matrix contains individual autocovariances on the diagonal and cross-covariances off the diagonal:

\[
\Gamma(h) = \mathbb{E} \left[ (x_{t+h} - \mu)(x_t - \mu)' \right] = \begin{pmatrix}
\gamma_{1,1}(h) & \gamma_{1,2}(h) & \cdots & \gamma_{1,p}(h) \\
\gamma_{2,1}(h) & \gamma_{2,2}(h) & \cdots & \gamma_{2,p}(h) \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{p,1}(h) & \gamma_{p,2}(h) & \cdots & \gamma_{p,p}(h)
\end{pmatrix}
\]
Sample mean and autocovariances

• sample mean:

\[ \bar{x} = \frac{1}{n} \sum_{t=1}^{n} x_t \]

• sample autocovariance:

\[ \hat{\Gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} \left( x_{t+h} - \bar{x} \right) \left( x_t - \bar{x} \right)' \]

for \( h \geq 0 \), and \( \hat{\Gamma}(-h) = \hat{\Gamma}(h)' \).
Multidimensional Series (Spatial Statistics)

- Some studies involve data indexed by more than one variable.
- E.g. soil surface temperatures in a field
- Notation: $x_s$ is the observed value at location $s$ (s for spatial).
Soil temperatures

![3D graph of soil temperatures with temperature on the y-axis, columns on the x-axis, and rows on the z-axis. The graph shows fluctuations in temperature across the grid.]
Autocovariance and Variogram

- **Stationary**: \( E(x_s) \) and \( \text{cov}(x_{s+h}, x_s) \) do not depend on \( s \).

- For a stationary process, the autocovariance function is
  \[
  \gamma(h) = \text{cov}(x_{s+h}, x_s) = E \left[ (x_{s+h} - \mu)(x_s - \mu) \right]
  \]

- **Intrinsic**: \( E(x_{s+h} - x_s) \) and \( \text{var}(x_{s+h} - x_s) \) do not depend on \( s \).

- For an intrinsic process, the (semi-)variogram is
  \[
  V_x(h) = \frac{1}{2} \text{var}(x_{s+h} - x_s)
  \]
• A stationary process is intrinsic (see Problem 1.26), but an intrinsic process is not necessarily stationary.

• In one dimension, the random walk is intrinsic but not stationary.

• When stationary, $V_x(h) = \gamma(0) - \gamma(h)$.

• *Isotropic:* an intrinsic process is isotropic if the variogram is a function only of $|h|$, the Euclidean distance between $s + h$ and $s$. 