Volatility in Several Series

- The various xARCH models provide many ways to model volatility dynamics in a single series.
- Many areas of financial risk involve the joint behavior of several variables.
- Graphs show that volatility typically varies simultaneously across series.
- Individual xARCH models could be fitted to each variable, but would not be linked.
GARCH

- Recall the GARCH model:

  \[ y_t = \text{financial variable, such as log-return on some asset} \]

  or perhaps the residual of some variable from an ARIMA model for the conditional mean structure.

- Assume \( y_t = \sigma_t \epsilon_t \), where \( \{\epsilon_t\} \) are independent and follow some fixed distribution (standard normal, standardized \( t \), \ldots).

- GARCH(1, 1):

  \[ \sigma_t^2 = \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2. \]

- The “standardized residuals” are \( z_t = \hat{\sigma}_t^{-1} y_t \).
Multivariate GARCH

* With several series,

\[ y_t = \text{vector of } J \text{ financial variables}. \]

* The analog of

\[ \sigma_t^2 = \text{var}(y_t | y_{t-1}, y_{t-2}, \cdots) \]

is

\[ \Sigma_t = \text{var}(y_t | y_{t-1}, y_{t-2}, \cdots). \]

* The purpose of a multivariate xARCH model is to provide a recursive expression for \( \Sigma_t \).
PC-GARCH: use PCA.

First stage: fit univariate GARCH models to the individual series.

Put standardized residuals for time $t$ in a vector

$$ z_t = S_{1,t}^{-1} y_t, $$

where $S_{1,t}$ is the diagonal matrix of first stage conditional standard deviations.

Set up a data matrix and use the SVD:

$$ Z = (T \times J) \text{ data matrix with rows } z'_t, t = 1, 2, \ldots, T $$

$$ = \begin{pmatrix} U \\ T \times J \end{pmatrix} \begin{pmatrix} D \\ J \times J, \text{diagonal} \end{pmatrix} \begin{pmatrix} V' \\ J \times J \end{pmatrix} $$
Second stage: fit univariate GARCH models to the PC scores (here columns of $U$; more conventionally, of $UD$).

Write

$$\epsilon_t = S^{-1}_{2,t} u_t$$

where $S_{2,t}$ is the diagonal matrix of second stage conditional standard deviations.

Model $\epsilon_t$ as:

- $N_J(0, I)$;
- $t_{J,\nu}(0, I)$;
- a meta $t$-distribution, combining marginal $t$-distributions and a $t$-copula, all potentially with different degrees of freedom;
- some other non-Gaussian distribution with chosen tail length, tail dependence, and shape.
The conditional distribution of $y_{T+1}$ is represented as

$$y_{T+1} = S_{1,T+1} z_{T+1}$$

$$= S_{1,T+1} V D u_{T+1}$$

$$= S_{1,T+1} V D S_{2,T+1} \epsilon_{T+1}$$

where:

- $V$ and $D$ come from the SVD;
- $S_{1,T+1}$ and $S_{2,T+1}$ come from the first and second stage GARCH recursions, respectively;
- $\epsilon_{T+1}$ follows the chosen model.

Note that the distribution of $\epsilon_{T+1}$ does not depend on the past, so it is independent of the past.
If the chosen distribution of $\epsilon_{T+1}$ is Gaussian or $t$, the conditional distribution of $y_{T+1}$ is in the same family.

Otherwise, the conditional distribution of $y_{T+1}$ is likely to be intractable except for simulation.

Either way, use it for instance to compute VaR or ES of a portfolio of the assets on which these are the returns.
Reduced Rank PC-GARCH: recall that if \( d_1^2 + d_2^2 + \cdots + d_k^2 \gg d_{k+1}^2 + \cdots + d_J^2 \) then

\[
Z \approx \begin{pmatrix}
U^{(k)} \\
T \times k
\end{pmatrix}
\begin{pmatrix}
D^{(k)} \\
k \times k, \text{diagonal}
\end{pmatrix}
\begin{pmatrix}
V^{(k)'} \\
k \times J
\end{pmatrix}.
\]

So only \( k \) principal component score series need to be modeled, and \( \epsilon_t \) consists of only \( k \) variables whose distributions need to be modeled.

If \( k \ll J \), modeling is much simplified.
PC-GARCH and Reduced Rank PC-GARCH are related to Orthogonal GARCH (OGARCH) and Generalized Orthogonal GARCH (GO-GARCH).

These model the conditional covariance matrix of \( y_t \) directly:

\[
\Sigma_t = XD_tX'
\]

where \( D_t \) is a diagonal matrix of univariate GARCH conditional variances, and:

- in OGARCH, \( X \) is \((J \times k)\) with orthogonal columns, like \( V^{(k)} \);
- in GO-GARCH, \( X \) is \((J \times J)\) with no orthogonality constraints.

Then \( y_t \) is modeled as Gaussian or \( t \) with covariance matrix \( \Sigma_t \).

Note: PC-GARCH makes \( \Sigma_t \) more complicated:

\[
\Sigma_t = S_{1,t} V D S^2_{2,t} D V' S_{1,t}.
\]
Non-PCA Approaches

- **MGARCH** is a very general extension of univariate GARCH.
- **MGARCH(1, 1):**

\[
\text{vech}(\Sigma_t) = A \text{vech}(y_{t-1}y_{t-1}') + B \text{vech}(\Sigma_{t-1}) + c
\]

where vech(·) vectorizes the lower triangle of a symmetric matrix:

\[
\text{vech}(S) = \begin{bmatrix}
S_{1,1} \\
S_{2,1} \\
\vdots \\
S_{J,1} \\
S_{2,2} \\
\vdots \\
S_{J,J}
\end{bmatrix}
\]
MGARCH is over-parametrized: $\text{vech}(\mathbf{S})$ is $(J(J + 1)/2 \times 1)$. So $\mathbf{A}$ and $\mathbf{B}$ are $((J(J + 1)/2) \times (J(J + 1)/2))$, with $\sim J^4/4$ entries each, and $\mathbf{c}$ is $((J(J + 1)/2) \times 1)$. Constraining $\Sigma_t$ to be non-negative definite is a challenge.
In **diag-MGARCH**, \( A \) and \( B \) are constrained to be diagonal—some improvement.

Note: diagonal multiplication of \( \text{vech}(\cdot) \) is equivalent to entrywise multiplication:

\[
\Sigma_t = A \circ (y_{t-1}y_{t-1}') + B \circ \Sigma_{t-1} + C
\]

where \( A \), \( B \), and \( C \) are now \((J \times J)\), and “\( \circ \)” denotes entrywise (Hadamard, or Schur) product.

Constraining \( \Sigma_t \) to be non-negative definite is still a challenge: requiring \( A \), \( B \), and \( C \) to be non-negative definite is sufficient, but not necessary.

No “cross-talk” in diag-MGARCH.
BEKK is a different simplified form of MGARCH:

$$\Sigma_t = A'(y_{t-1}y'_{t-1})A + B'\Sigma_{t-1}B + C$$

Here $A$ and $B$ are unrestricted, and $C$ is positive definite symmetric.

Off-diagonal entries in $A$ and $B$ introduce cross-talk: volatility in one variable can flow into another from either the $y_{t-1}y'_{t-1}$ term or the $\Sigma_{t-1}$ term.
Constant Conditional Correlation (CCC)

- We can always decompose $\Sigma_t$:

$$\Sigma_t = D_t R_t D_t,$$

where $D_t$ is the diagonal matrix of conditional standard deviations, and $R_t$ is the conditional correlation matrix.

- In CCC, assume that $R_t = R$, constant.

- Use separate GARCH models to build $D_t$, and estimate $R$ from the standardized residuals $z_t$. 
Dynamic Conditional Correlation (DCC)

- As in CCC, use separate GARCH models to build $D_t$.
- Then

$$R_t = (\text{diag}(Q_t))^{-1/2} Q_t (\text{diag}(Q_t))^{-1/2}$$

where $Q_t$ satisfies the recursion

$$Q_t = \theta_1 z_{t-1} z_{t-1}' + \theta_2 Q_{t-1} + (1 - \theta_1 - \theta_2) \bar{Q}.$$  

That is, $Q_t$ follows an even simpler MGARCH model with scalar multipliers, but driven by the standardized residuals $z_t$ instead of the original returns $y_t$.

- $R_t$ simply extracts the correlation structure from $Q_t$. 

Role of the Copula

- In each case, we construct
  \[ \Sigma_t = \text{var}(y_t | y_{t-1}, y_{t-2}, \ldots) \].

- Write
  \[ \epsilon_t = \Sigma_t^{-1/2} y_t \]
  so that
  \[ \text{var}(\epsilon_t) = I_J. \]

- Here \( \Sigma_t^{-1/2} \) could be any inverse square root of \( \Sigma_t \); e.g. triangular (Choleski), symmetric positive definite (spectral decomposition).

- In PC-GARCH, a specific version was implied.
The distribution of $\epsilon_t$ might be assumed to be independent normal (with no tail dependence), or multivariate $t$ (with positive tail dependence).

It could alternatively be constructed to have appropriate tail lengths and appropriate tail dependences by separately estimating:

- the marginal distribution of each component of $\epsilon_t$;
- a copula to introduce nonlinear dependence.

Note: if $\Sigma_t^{-1/2}$ is non-sparse, the tail properties of components of $\epsilon_t$ affect all the components of $y_t$.

- The symmetric positive definite version relates each component of $y_t$ most closely to the corresponding component of $\epsilon_t$. 