Section 1

Statistics of Market Rates, continued
Yield Curves

Figure 1 shows one view of the US yield curve from the end of 1996 to early August, 2003. The set of maturities is: 3mo; 6mo; 1yr; 2yr; 3yr; 5yr; 7yr; 10yr; 15yr; 30yr.

Figure 1: US yield curve.
Figure 2 shows another view of the same yields.

Figure 2: Perspective view of US yield curve.
The yields for maturities of a year or less are LIBOR. Data for longer maturities are “swaps curve” data, a version of the yield curve that is adapted for use in an interest rate swap (IRS). The entry on a given date and for a given maturity is the interest rate for the fixed side of an IRS, as set by the market on that date. That is, an entity wishing to begin an IRS, fixed versus LIBOR flat, would have been quoted this rate for the fixed side, if the swap was to have been entered “at the market”.

The rates are averages of quotes from a number of dealers. These are “mid-market” rates, that is the average of the “bid” (lower) and “ask” (higher) rates. The difference between the bid and ask rates is the bid-ask spread, typically around 2 basis points, or 0.02%, for USD swaps.
Because the fixed side of an IRS has periodic payments, the swaps curve is a \textit{current coupon} yield curve rather than a \textit{zero coupon} curve. However, as we have noted elsewhere, one may be obtained from the other without difficulty. The swaps curve is typically 20-30 basis points higher than the Treasury yield curve, reflecting the presence of a small amount of risk that is absent in Treasury securities.
Statistical treatment

A yield curve may be thought of as a multivariate random variable. One obvious summary of data such as these is through principal components analysis (PCA). Figure 3 shows the loadings for the first component, which accounts for 96.4% of the total variance. They are graphed against the square root of maturity, for graphical clarity.

**Figure 3:** Loadings for first PC of US yields.
Figure 4 shows the loadings for the second component, which accounts for 3.4% of the total variance.

![Figure 4: Loadings for second PC of US yields.](image-url)
The loadings for the first component are all positive, and therefore represent *shifts*: a tendency for yields to move in tandem across all maturities. The loadings are all of similar magnitude for maturities of a year or more, and to this extent represent *parallel* shifts. The smaller loadings at shorter maturities imply that these rates are less strongly affected by this dominant mode of variation.
The second set of loadings is negative at the short end of the curve, and positive at the long end, and represents *tilts*: a tendency for the slope of the yield curve to vary. Note that these tilts would be *linear* only on a time scale on which the 10-year and 30-year rates are very close together.
It may be more useful to look at the yield curves in terms of day-to-day changes, as it generally is for foreign exchange rate data. The first two components account for 85.1% and 9.5% of variance in this case (the third accounts for 3.1%). The loadings for the first component, still broadly interpretable as a shift factor, are shown in Figure 5.

![Figure 5](image)

**Figure 5**: Loadings for first PC of day-to-day changes in US yields.
The loadings for the second component are shown in Figure 6. These loadings are still broadly interpretable as a tilt factor, but evidently the behavior is not as simple as when studying the levels of yields themselves.

Figure 6: Loadings for second PC of day-to-day changes in US yields.
Note in particular that there appears to be a break in behavior between maturities of one year and less versus the longer maturities. This reflects the different markets that determine LIBOR rates and swap rates.

In both pairs of loadings, it would presumably be possible to find linear combinations that more closely represent parallel shifts and linear tilts, respectively. These would jointly capture the same total fraction of the overall variance, but because of a loss of orthogonality it would no longer be possible to distribute this fraction unambiguously between them. This minor inconvenience might be a worthwhile price to pay for a more conventional interpretation of the decomposition.
Looking forward: modeling and simulation

The analyses of the previous section suggest that a statistical model for yield curves might be based on a low-dimensional representation derived from PCA or a similar method. Each component is associated with a time series of scores, which would need to be modeled in the same general way as was discussed earlier for FX rates.
There are, however, constraints that should be imposed. In the case of FX rates, the only constraint is that they should be positive, and this is easily imposed by modeling the log rate. Interest rates are also intrinsically positive, again suggesting modeling on the log scale. Swaps payments may of course be negative, for instance if they are set according to a floating rate less a spread, unless there is a floor (express or implied) at zero.
Interest rates however also satisfy some other constraints. The zero-coupon yield curve is related to the discount curve by $D(t) = e^{-tr(t)}$, and it is natural for $D(t)$ to be required to be monotonically decreasing. One way to represent this is in terms of the implied forward rates. For $0 < t < u$, write

$$r(t, u) = \frac{\log D(u) - \log D(t)}{u - t}$$

whence

$$D(u) = D(t)e^{-(u-t)r(t,u)}.$$
The quantity \( r(t, u) \) is referred to as the forward rate at time \( t \) for a loan maturing at time \( u \). A deposit that earned the rate \( r(t) \) until time \( t \) and was then deposited for the additional time \( u - t \) at a rate of \( r(t, u) \) would accumulate to the same value as if it were deposited until time \( u \) at the current rate for such deposits, \( r(u) \). This is the sense in which the rate is \textit{implied} for forward deposits. It is then clear that the \textit{discount curve is monotonically decreasing if and only if all implied forward rates are positive}. 
It is also clear that any of the sets of data:

- $D(t_1), D(t_2), \ldots, D(t_n)$;
- $r(t_1), r(t_2), \ldots, r(t_n)$;
- $r(t_1), r(t_1, t_2), \ldots, r(t_{n-1}, t_n)$;

may be obtained from any other, and amount to alternative representations of the same yield curve. Since the last satisfy the simplest constraints, modeling and simulating the logarithms of the implied forward rates may be the most direct way of incorporating the monotonicity property.
Figure 7 shows the implied forward rates corresponding to the rates shown in Figure 1. While they are different from the original yields, there is a broad similarity, and PCA also shows broadly similar results.

Figure 7: US implied forwards.
The loadings for the first component are shown in Figure 8. As before, the loadings are all positive, although they range in magnitude.

**Figure 8**: Loadings for first PC of day-to-day changes in log forward rates.
Figure 9 shows the corresponding time series component.

**Figure 9**: Time series for first PC of day-to-day changes in log forward rates.
The graph reveals two features that may require modelling:

- in some time intervals, the changes appear to be centered around non-zero values; if the serial correlations are significant, a conditional mean (ARMA) model may be needed;
- the volatility is clearly not constant, and a conditional variance model is needed, perhaps for the residuals from an ARMA conditional mean model.
However, the conditional variance structure does not necessarily mean that we should fit a CH model. The increase in volatility in late 2001 coincides with a progressive decline in short-term rates. If the noise in the rates were roughly constant, we would expect the noise in their logarithms to increase as the rate decrease, as logarithmic changes are approximately the same as proportional changes. In this case, we might model

\[ x_t = r_{t-1}(t_{i-1}, t_i) \log[r_t(t_{i-1}, t_i)/r_{t-1}(t_{i-1}, t_i)] \]

instead of the first usual first difference

\[ \log[r_t(t_{i-1}, t_i)] - \log[r_{t-1}(t_{i-1}, t_i)] = \log[r_t(t_{i-1}, t_i)/r_{t-1}(t_{i-1}, t_i)]. \]

We refer to this \( x_t \) as a \textit{scaled} difference.
The loadings for the first component for the scaled differences are shown in Figure 10. These loadings are qualitatively similar to those for the day-to-day changes in yields themselves (Figure 5).

![Figure 10](image)

*Figure 10:* Loadings for first PC of scaled differences of log forward rates.
Figure 11 shows the corresponding time series component. Changes in volatility are much smaller than those in Figure 9, but are still visible.

**Figure 11**: Time series for first PC of scaled differences log forward rates.
The result of fitting a first-order autoregression to the time series is shown in Figure 12

Call:
arma(x = dlfwdfwdSvd$u[, 1], order = c(1, 0, 0))

Model:
ARMA(1,0)

Residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1.146e-01</td>
<td>-1.242e-02</td>
<td>6.836e-05</td>
<td>1.458e-02</td>
<td>9.278e-02</td>
</tr>
</tbody>
</table>

Coefficient(s):

|     | Estimate | Std. Error | t value | Pr(>|t|) |
|-----|----------|------------|---------|---------|
| ar1 | 0.0701904| 0.0240048  | 2.924   | 0.00346 ** |
| intercept | 0.0003400 | 0.0005766 | 0.590   | 0.55540 |

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Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

Fit:
sigma^2 estimated as 0.000575, Conditional Sum-of-Squares = 0.99, AIC = -7989.65

Figure 12: AR(1) model for first component time series.
The autoregressive coefficient $a_1$ is small (0.07) but quite significant ($P = 0.35\%$). The corresponding residuals are shown in Figure 13, and are somewhat more zero-centered than the series in Figure 11.

![Figure 13: Residuals from ARMA fit.](image)
Figures 14 and 15 show the GARCH(1, 1) fit to these residuals.

Call:
garch(x = a)

Model:
GARCH(1,1)

Residuals:

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>1Q</th>
<th>Median</th>
<th>3Q</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.972</td>
<td>-0.559</td>
<td>0.00336</td>
<td>0.622</td>
<td>4.362</td>
</tr>
</tbody>
</table>

Coefficient(s):

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|---------|
| a0        | 9.812e-06  | 3.427e-06 | 2.863  | 0.00419 ** |
| a1        | 5.952e-02  | 9.120e-03 | 6.527  | 6.72e-11 *** |
| b1        | 9.255e-01  | 1.214e-02 | 76.216 | < 2e-16 *** |

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Signif. codes:  0 *** 0.01 ** 0.05 . 0.1  1

Figure 14 :  GARCH model for ARMA residuals.
Diagnostic Tests:
Jarque Bera Test

data: Residuals
X-squared = 116.2346, df = 2, p-value < 2.2e-16

Box-Ljung test

data: Squared.Residuals
X-squared = 0.39, df = 1, p-value = 0.5323

Figure 15: GARCH model for ARMA residuals, continued.

The recursion parameter b1 is 0.93, again indicating a long time scale for volatility, and is highly significant, even though the variations in volatility are much smaller than in Figure 9.
The $qq$-plot of the residuals from this GARCH fit versus the Gaussian distribution is shown in Figure 16. The general curvature indicates skewness, and a slight ogive shape suggests long tails.

![Normal Q–Q Plot](image)

**Figure 16:** Gaussian $qq$-plot of GARCH residuals.
The corresponding qq-plot against the t-distribution with 15 degrees of freedom, Figure 17, seems straighter overall.

Figure 17: qq-plot of GARCH residuals against t-distribution with 15 degrees of freedom.
In summary, the first component time series for the scaled differences could plausibly be modeled as a first order autoregression with $t15$-GARCH(1, 1) residuals.