Back to Topic 2: The “Failures” of Risk Management

- Financial risk is the potential for financial loss.

- Recall: we write $V(t)$ for the value of a financial portfolio on day $t$, with $t = 0$ representing the “present”.

- With a holding period of $h$ days, risk is therefore the potential for $V(h) - V(0)$, the change in value over the holding period, to be negative.
If, for instance:

- the portfolio is simply an investment in (a fund that tracks) the S&P 500 stock index
- and the holding period $h = 1$;

then we can use a fitted ARCH (or GARCH-\(t\), or whatever) model to state the conditional distribution of $V(h) - V(0)$, given the history of the index up to the present ($t \leq 0$).

We can use that conditional distribution to find the level of loss that will be exceeded with a chosen probability.

But is that a good measure of risk?
Coherent Risk Measures

- A theory of risk measures has been built up over the last 10+ years.

- It considers various portfolios, with payoffs $X, Y, \ldots$.

- A risk measure $m(\cdot)$ associates a real-valued risk quantity with any portfolio payoff.

- Note: in practice, $m(\cdot)$ is defined by a probability distribution for the random variables $X, Y, \ldots$, but the theory does not require one.
A risk measure is called coherent if it satisfies four criteria.

**Subadditivity:** For any portfolio payoffs $X$ and $Y$,

$$m(X + Y) \leq m(X) + m(Y);$$

**Homogeneity:** For any $k > 0$, $m(kX) = km(X)$;

**Monotonicity:** $m(X) \geq m(Y)$ if $X \leq Y$;

**Risk-free condition:** For any $k$, $m(X + k) = m(X) - k$. 
• Of these, Homogeneity is debatable: if $\alpha$ is large enough, the risk in the position may become unacceptable; however, it is innocuous for moderate values.

• Subadditivity is necessary to recognize diversification: combining two portfolios cannot increase the combined risk.
Scenarios

• One way to construct a risk measure is using *scenarios*.

• In the S&P 500 example, two scenarios might be:
  - the index goes up 2% (logarithmic return = 0.02);
  - the index goes down 2% (logarithmic return = −0.02).

• Find the loss (or profit) for $X$ in each scenario.

• Take $m(X)$ to be the greater of the losses across the scenarios.
Instead of hard scenarios like these, we could use probabilistic scenarios:

- logarithmic return $\sim N(0.02, \sigma^2)$;
- logarithmic return $\sim N(-0.02, \sigma^2)$.

Find the expected value $E(X)$ under each scenario.

We could then let

$$m(X) = \max [E_{+0.02}(X), E_{-0.02}(X)].$$
• The hard scenarios are special cases of probabilistic scenarios in which all random variables have degenerate distributions, taking a given value with probability 1.

• For a more complex portfolio, we would need more scenarios, each described by a joint probability distribution for all the portfolios \( X, Y, \ldots \).

• Write \( Q \) for a set of such distributions, and \( m_Q(\cdot) \) for the corresponding risk measure:

\[
m_Q(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X).
\]
• The basic result about coherent risk measures is:

A risk measure $m(\cdot)$ is coherent if and only if it is $m_Q(\cdot)$ for some family $Q$ of probabilistic scenarios.

• $\text{VaR}_\alpha$ (Value at Risk for confidence level $1 - \alpha$) is not coherent.
• For example:

  – $X$ and $Y$ are independent;

  – each takes the values

       \[
       \begin{cases}
       -a & \text{with probability } p \\
       0 & \text{with probability } 1 - p,
       \end{cases}
       \]

  where $a > 0$ and $\alpha/2 < p < \alpha$. Then

       \[
       \text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y) = 0,
       \]

  but

       \[
       \text{VaR}_\alpha(X + Y) = a.
       \]
• To find a better (coherent) risk measure, what family $Q$ should we use?

  – Scenarios are arbitrary, and only loosely based on historical patterns.

  – Statistical modeling of historical data gives us one probability distribution $P$ for what will happen between $t = 0$ and $t = h$, not a family.

• One solution: use *boundedly stressed* modifications of $P$. 
• A distribution $Q$ is an $\alpha$-stressed modification of $P$ if for every event $A$,

$$Q(A) \leq \frac{P(A)}{\alpha}.$$ 

• If $A$ is an event in which $X$ loses money, it is more probable under $Q$ than under $P$, but only boundedly so.

• If $\alpha$ is either of the common values .05 or .01, the multiplier is 20 or 100, respectively.
• Some interesting math shows that the resulting risk measure, necessarily coherent, is the *Expected Shortfall*, $\text{ES}_\alpha(X)$.

• If, under $P$, $X$ is continuously distributed with a positive density, then $\text{VaR}_\alpha(X)$ is the negative of the $\alpha$-quantile of $X$; that is, the (unique) solution $q$ to

$$P(X \leq -q) = \alpha.$$  

• Then

$$\text{ES}_\alpha(X) = E \left[ (-X) \mid (-X) \geq \text{VaR}_\alpha(X) \right].$$
• Since we are conditioning on \((-X) \geq \text{VaR}_\alpha(X)\), clearly
\[
\text{ES}_\alpha(X) \geq \text{VaR}_\alpha(X).
\]

• In fact,
\[
\text{ES}_\alpha(X) = \text{VaR}_\alpha(X) + \mathbb{E} \left[ \left( -X - \text{VaR}_\alpha(X) \right) \mid (-X) \geq \text{VaR}_\alpha(X) \right]
\]
\[
= \text{VaR}_\alpha(X) + \left( \frac{1}{\alpha} \right) \mathbb{E} \left\{ \max \left[ \left( -X - \text{VaR}_\alpha(X) \right), 0 \right] \right\}.
\]

• Note that if we bought insurance against loss, with a deductible of \(\text{VaR}_\alpha(X)\), the actuarially fair premium is
\[
\mathbb{E} \left\{ \max \left[ \left( -X - \text{VaR}_\alpha(X) \right), 0 \right] \right\} = \alpha \left[ \text{ES}_\alpha(X) - \text{VaR}_\alpha(X) \right].
\]

• See an opinion piece in the Financial Times.