Some Special Cases

- Recall: we can classify models:

  \[
  \text{mean} = \begin{cases} 
  \text{constant} \\
  \text{linear in } x_{u-1}, x_{u-2}, \ldots, x_{u-p} \\
  \text{other}
  \end{cases}
  \]

  \[
  \text{variance} = \begin{cases} 
  \text{constant} \\
  \text{linear in } x^2_{u-1}, x^2_{u-2}, \ldots, x^2_{u-p} \\
  \text{other}
  \end{cases}
  \]
Autoregressions

- The simplest special case is the autoregression.

- If:

\[ \mu_t = \text{linear function of } x_{t-1}, x_{t-2}, \ldots, x_{t-p} \]
\[ = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} \]

and

\[ \sigma_t^2 = \text{constant} \]
\[ = \tau^2 \]

then \( X_t \) is called autoregressive of order \( p \) (AR(\( p \))).
\begin{itemize}
  \item We usually define
  \begin{align*}
    \epsilon_t &= X_t - \mathbb{E}(X_t \mid X_{t-1}, X_{t-2}, \ldots) \\
    &= X_t - \left( \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} \right).
  \end{align*}

  \item We then write the model as
  \begin{align*}
    X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t
  \end{align*}
  where
  \begin{align*}
    \mathbb{E}(\epsilon_t \mid X_{t-1}, X_{t-2}, \ldots) &= 0, \\
    \text{and}
    \text{Var}(\epsilon_t \mid X_{t-1}, X_{t-2}, \ldots) &= \tau^2.
  \end{align*}
\end{itemize}
• If in addition the shape of the conditional density is fixed, then the $\epsilon$’s are independent and identically distributed.

• The polynomial

$$\phi(z) = 1 - \left( \phi_1z + \phi_2z^2 + \cdots + \phi_pz^p \right),$$

where we view $z$ as a complex variable, plays an important role in the theory of autoregressions.
• In particular, if the zeros of $\phi(z)$ are outside the unit circle, then:

- $1/\phi(z)$ has a Taylor series expansion

$$\frac{1}{\phi(z)} = 1 + \psi_1 z + \psi_2 z^2 + \ldots$$

which converges for $|z| \leq 1$;

- $X_t$ is a linear combination of $\epsilon_t, \epsilon_{t-1}, \ldots$:

$$X_t = \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \ldots$$
• These equations are often written in terms of the \textit{back-shift operator} $B$, defined by
\[ BX_t = X_{t-1}, \quad B\epsilon_t = \epsilon_{t-1}, \quad \ldots \]

• Then
\[
\epsilon_t = X_t - \left( \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} \right) \\
= X_t - \left( \phi_1 BX_t + \phi_2 B^2 X_t + \cdots + \phi_p B^p X_t \right) \\
= \phi(B)X_t.
\]
• So it is natural to write

\[ X_t = \frac{1}{\phi(B)} \epsilon_t \]

\[ = \left( 1 + \psi_1 B + \psi_2 B^2 + \ldots \right) \epsilon_t \]

\[ = \epsilon_t + \psi_1 \epsilon_{t-1} + \psi_2 \epsilon_{t-2} + \ldots \]
Conditional Heteroscedasticity: ARCH

- Another special case is Engle’s AutoRegressive Conditionally Heteroscedastic, or ARCH, model.

- If

  \[ \mu_t = \text{constant} \]

  and

  \[ \sigma_t^2 = \text{linear function of } x_{t-1}^2, x_{t-2}^2, \ldots, x_{t-q}^2 \]

  \[ = \omega + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \cdots + \alpha_q x_{t-q}^2 \]

  then \( X_t \) is called AutoRegressive Conditionally Heteroscedastic of order \( q \) (ARCH(\( q \))).
ARCH models are important in finance, because many financial time series show variances that fluctuate over time, while usually having constant, essential zero, conditional means.
A Modest Generalization

- In practice, large values of $p$ or $q$ are sometimes needed to get a good fit with the AR($p$) and ARCH($q$) models.

- That introduces many parameters to be estimated, which is problematic.

- We need models that allow large $p$ with few parameters.
• Suppose for instance that

\[ X_t = \theta X_{t-1} - \theta^2 X_{t-2} - \cdots + \epsilon_t \]

for some \( \theta, -1 < \theta < 1 \).

• That is, \( p = \infty \), but \( \phi_r = -(-\theta)^r \Rightarrow \) only one parameter, \( \theta \).

• Then

\[ \epsilon_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} + \cdots = \frac{1}{1 + \theta B} X_t, \]

or

\[ X_t = (1 + \theta B)\epsilon_t = \epsilon_t + \theta \epsilon_{t-1}. \]
• This is called a *Moving Average* model; specifically, the first-order Moving Average, MA(1).

• The general MA($q$) model has $q$ terms:

\[
X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} \\
= (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q) \epsilon_t \\
= \theta(B) \epsilon_t.
\]
• We can mix AR and MA structure:

\[ X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} \]

or

\[ X_t - \left( \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} \right) = \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \cdots + \theta_q \epsilon_{t-q} \]

or

\[ \phi(B) X_t = \theta(B) \epsilon_t. \]

• This is the AutoRegressive Moving Average model of order \((p, q)\) (ARMA\((p, q)\)).
Integrated Models

- Sometimes we cannot find an ARMA($p,q$) that fits the data for reasonably small order $p$ and $q$.

- For instance, a random walk

$$X_t = X_{t-1} + \epsilon_t$$

is like an AR(1) but with $\phi = 1$.

- But the \textit{first differences} $X_t - X_{t-1}$ are simple:

$$X_t - X_{t-1} = \epsilon_t$$

a trivial model with $p = q = 0$. 
• More generally, we might find that $X_t - X_{t-1}$ is ARMA($p, q$).

• More generally yet, we might have to difference $X_t$ more than once.

• Note that

$$X_t - X_{t-1} = (1 - B)X_t.$$ 

• Define the $d^{th}$ difference by $(1 - B)^dX_t$, $d = 1, 2, \ldots$.

• If the $d^{th}$ difference of $X_t$ is ARMA($p, q$), we say that $X_t$ is AutoRegressive Integrated Moving Average of order ($p, d, q$) (ARIMA($p, d, q$)).