ARMA Autocorrelation Functions

- For a moving average process, MA($q$):
  \[ x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}. \]

- So (with $\theta_0 = 1$)
  \[
  \gamma(h) = \text{cov}(x_{t+h}, x_t) = \sum_{j=0}^{q} \theta_j \sum_{k=0}^{q} \theta_k
  = \begin{cases} 
  \sigma^2_w \sum_{j=0}^{q-h} \theta_j \theta_{j+h}, & 0 \leq h \leq q \\
  0, & h > q.
  \end{cases}
  \]
• So the ACF is

\[
\rho(h) = \begin{cases} 
q-h \sum_{j=0}^{q} \theta_j \theta_{j+h} & \text{if } 0 \leq h \leq q \\
q \sum_{j=0}^{q} \theta_j^2 & \text{if } h > q.
\end{cases}
\]

• Notes:

– In these expressions, \( \theta_0 = 1 \) for convenience.

– \( \gamma(q) \neq 0 \) but \( \gamma(h) = 0 \) for \( h > q \). This characterizes MA\((q)\).
• For an autoregressive process, AR($p$):

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t.$$ 

• So

$$\gamma(h) = \text{COV}(x_{t+h}, x_t)$$

$$= E \left[ \left( \sum_{j=1}^{p} \phi_j x_{t+h-j} + w_{t+h} \right) x_t \right]$$

$$= \sum_{j=1}^{p} \phi_j \gamma(h-j) + \text{COV} (w_{t+h}, x_t).$$
Because $x_t$ is causal, $x_t$ is $w_t + a$ linear combination of $w_{t-1}, w_{t-2}, \ldots$.

So

$$\text{cov} \left( w_{t+h}, x_t \right) = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & h > 0. \end{cases}$$

Hence

$$\gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j), \quad h > 0$$

and

$$\gamma(0) = \sum_{j=1}^{p} \phi_j \gamma(-j) + \sigma_w^2.$$
If we know the parameters $\phi_1, \phi_2, \ldots, \phi_p$ and $\sigma^2_w$, these equations for $h = 0$ and $h = 1, 2, \ldots, p$ form $p + 1$ linear equations in the $p + 1$ unknowns $\gamma(0), \gamma(1), \ldots, \gamma(p)$.

The other autocovariances can then be found recursively from the equation for $h > p$.

Alternatively, if we know (or have estimated) $\gamma(0), \gamma(1), \ldots, \gamma(p)$, they form $p + 1$ linear equations in the $p + 1$ parameters $\phi_1, \phi_2, \ldots, \phi_p$ and $\sigma^2_w$.

These are the Yule-Walker equations.
• For the ARMA($p, q$) model with $p > 0$ and $q > 0$:

\[ x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}, \]

a generalized set of Yule-Walker equations must be used.

• The moving average models ARMA$(0, q) = MA(q)$ are the only ones with a closed form expression for $\gamma(h)$.

• For AR($p$) and ARMA($p, q$) with $p > 0$, the recursive equation means that for $h > \max(p, q + 1)$, $\gamma(h)$ is a sum of geometrically decaying terms, possibly damped oscillations.
• The recursive equation is

\[ \gamma(h) = \sum_{j=1}^{p} \phi_j \gamma(h-j), \quad h > q. \]

• What kinds of sequences satisfy an equation like this?
  
  – Try \( \gamma(h) = z^{-h} \) for some constant \( z \).
  
  – The equation becomes

\[
0 = z^{-h} - \sum_{j=1}^{p} \phi_j z^{-(h-j)} = z^{-h} \left( 1 - \sum_{j=1}^{p} \phi_j z^j \right) = z^{-h} \phi(z).
\]
• So if \( \phi(z) = 0 \), then \( \gamma(h) = z^{-h} \) satisfies the equation.

• Since \( \phi(z) \) is a polynomial of degree \( p \), there are \( p \) solutions, say \( z_1, z_2, \ldots, z_p \).

• So a more general solution is

\[
\gamma(h) = \sum_{l=1}^{p} c_l z_l^{-h},
\]

for any constants \( c_1, c_2, \ldots, c_p \).

• If \( z_1, z_2, \ldots, z_p \) are distinct, this is the most general solution; if some roots are repeated, the general form is a little more complicated.
• If all $z_1, z_2, \ldots, z_p$ are real, this is a sum of geometrically decaying terms.

• If any root is complex, its complex conjugate must also be a root, and these two terms may be combined into geometrically decaying sine-cosine terms.

• The constants $c_1, c_2, \ldots, c_p$ are determined by initial conditions; in the ARMA case, these are the Yule-Walker equations.

• Note that the various rates of decay are the zeros of $\phi(z)$, the autoregressive operator, and do not depend on $\theta(z)$, the moving average operator.
• Example: ARMA(1, 1)

\[ x_t = \phi x_{t-1} + \theta w_{t-1} + w_t. \]

• The recursion is

\[ \gamma(h) = \phi \gamma(h-1), \quad h = 2, 3, \ldots \]

• So \( \gamma(h) = c\phi^h \) for \( h = 1, 2, \ldots \), but \( c \neq 1 \).

• Graphically, the ACF decays geometrically, but with a different value at \( h = 0 \).
The Partial Autocorrelation Function

- An MA($q$) can be identified from its ACF: non-zero to lag $q$, and zero afterwards.

- We need a similar tool for AR($p$).

- The partial autocorrelation function (PACF) fills that role.
• Recall: for multivariate random variables $X, Y, Z$, the partial correlations of $X$ and $Y$ given $Z$ are the correlations of:
  
  – the residuals of $X$ from its regression on $Z$; and
  
  – the residuals of $Y$ from its regression on $Z$.

• Here “regression” means conditional expectation, or best linear prediction, based on population distributions, not a sample calculation.

• In a time series, the partial autocorrelations are defined as

$$
\phi_{h,h} = \text{partial correlation of } x_{t+h} \text{ and } x_t
$$

given $x_{t+h-1}, x_{t+h-2}, \ldots, x_{t+1}$.
• For an autoregressive process, AR($p$):

\[ x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t, \]

• If $h > p$, the regression of $x_{t+h}$ on $x_{t+h-1}, x_{t+h-2}, \ldots, x_{t+1}$ is

\[ \phi_1 x_{t+h-1} + \phi_2 x_{t+h-2} + \cdots + \phi_p x_{t+h-p} \]

• So the residual is just $w_{t+h}$, which is uncorrelated with $x_{t+h-1}, x_{t+h-2}, \ldots, x_{t+1}$ and $x_t$. 
• So the partial autocorrelation is zero for $h > p$:

$$\phi_{h,h} = 0, \quad h > p.$$  

• We can also show that $\phi_{p,p} = \phi_p$, which is non-zero by assumption.

• So $\phi_{p,p} \neq 0$ but $\phi_{h,h} = 0$ for $h > p$. This characterizes AR($p$).
The Inverse Autocorrelation Function

- SAS’s proc arima also shows the Inverse Autocorrelation Function (IACF).

- The IACF of the ARMA($p,q$) model

  \[ \phi(B)x_t = \theta(B)w_t \]

  is defined to be the ACF of the inverse (or dual) process

  \[ \theta(B)x_t^{(\text{inverse})} = \phi(B)w_t. \]

- The IACF has the same property as the PACF: AR($p$) is characterized by an IACF that is nonzero at lag $p$ but zero for larger lags.
Summary: Identification of ARMA processes

- AR($p$) is characterized by a PACF or IACF that is:
  - nonzero at lag $p$;
  - zero for lags larger than $p$.

- MA($q$) is characterized by an ACF that is:
  - nonzero at lag $q$;
  - zero for lags larger than $q$.

- For anything else, try ARMA($p,q$) with $p > 0$ and $q > 0$. 
For $p > 0$ and $q > 0$:

<table>
<thead>
<tr>
<th></th>
<th>AR$(p)$</th>
<th>MA$(q)$</th>
<th>ARMA$(p, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACF</td>
<td>Tails off</td>
<td>Cuts off after lag $q$</td>
<td>Tails off</td>
</tr>
<tr>
<td>PACF</td>
<td>Cuts off after lag $p$</td>
<td>Tails off</td>
<td>Tails off</td>
</tr>
<tr>
<td>IACF</td>
<td>Cuts off after lag $p$</td>
<td>Tails off</td>
<td>Tails off</td>
</tr>
</tbody>
</table>

- Note: these characteristics are used to guide the initial choice of a model; estimation and model-checking will often lead to a different model.
Other ARMA Identification Techniques

- SAS’s `proc arima` offers the MINIC option on the `identify` statement, which produces a table of SBC criteria for various values of $p$ and $q$.

- The `identify` statement has two other options: ESACF and SCAN.

- Both produce tables in which the pattern of zero and non-zero values characterize $p$ and $q$.

- See Section 3.4.10 in Brocklebank and Dickey.
options linesize = 80;
ods html file = 'varve3.html';

data varve;
    infile '../data/varve.dat';
    input varve;
    lv = log(varve);
run;

proc arima data = varve;
    title 'Use identify options to identify a good model';
    identify var = lv(1) minic esacf scan;
    estimate q = 1 method = ml;
    estimate q = 2 method = ml;
    estimate p = 1 q = 1 method = ml;
run;

• proc arima output