Matrix Concepts

- Matrix $X = \text{rectangular array of numbers } (n \times p)$

- Vector $x = \text{single column } = n \times 1 \text{ matrix}$

- Length, direction, angle

- Linear dependence of vectors $x_1, x_2, \ldots, x_k$:
  
  $$c_1x_1 + c_2x_2 + \ldots c_kx_k = 0$$

  for some $c_1, c_2, \ldots, c_k$ not all zero.
• Transposed matrix $A' = B$: $b_{i,j} = a_{j,i}$

• Symmetric matrix: $A' = A$

• Matrix addition $A + B$, scalar multiplication $cA$

• Matrix product $AB$

• Identity matrix $I$: $IA = AI = A$

• Matrix inverse $A^{-1}$: $A^{-1}A = AA^{-1} = I$; exists only for square $A$ with linearly independent rows/columns
• Orthogonal matrix $Q$: $Q^{-1} = Q'$

• Eigenvalue $\lambda$, eigenvector $x$: $Ax = \lambda x$

• Square $k \times k$ symmetric $A$ has
  
  $-$ $k$ eigenvalue/vector pairs $(\lambda_j, e_j)$

  $-$ eigenvectors may be made orthonormal:

  $$e'_j e_k = \begin{cases} 
  1 & j = k \\
  0 & j \neq k 
  \end{cases}$$
• Positive definite symmetric $A$: $x'Ax > 0$ for any $x \neq 0$

• Positive definite $A$ has all positive eigenvalues.

• Spectral decomposition of a $(k \times k)$ symmetric matrix

\[ A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' + \ldots + \lambda_k e_k e_k' \]
• Note: the spectral decomposition may also be written

\[ A = P\Lambda P' \]

where:

– the columns of \( P \) are \( e_1, e_2, \ldots \):

\[ P = [e_1 e_2 \ldots e_k] ; \]

– consequently \( P \) is orthogonal;

– \( \Lambda \) is the diagonal matrix

\[ \Lambda = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_k
\end{bmatrix} \]
Recall:

- Spectral decomposition of a \((k \times k)\) symmetric matrix

\[
A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2' + \cdots + \lambda_k e_k e_k'.
\]

- Then

\[
A^2 = A \times A = \lambda_1^2 e_1 e_1' + \lambda_2^2 e_2 e_2' + \cdots + \lambda_k^2 e_k e_k'.
\]

- More generally,

\[
A^n = A \times A \times \cdots \times A = \lambda_1^n e_1 e_1' + \lambda_2^n e_2 e_2' + \cdots + \lambda_k^n e_k e_k'.
\]
• If $A$ is non-negative definite, it has a square root:

$$A^{1/2} = \sqrt{\lambda_1}e_1e_1' + \sqrt{\lambda_2}e_2e_2' + \cdots + \sqrt{\lambda_k}e_ke_k'.$$

• $A^{1/2}$ is also symmetric and non-negative definite,

• and of course

$$A^{1/2}A^{1/2} = \left(A^{1/2}\right)^2 = A.$$
• If $A$ is also \textit{positive} definite, it has an inverse:

$$A^{-1} = \frac{1}{\lambda_1} e_1 e_1' + \frac{1}{\lambda_2} e_2 e_2' + \cdots + \frac{1}{\lambda_k} e_k e_k'$$

• $A^{-1}$ is also symmetric and positive definite,

• and of course

$$A^{-1}A = AA^{-1} = I.$$