Comparing Means

• Many statistical analyses involve comparing mean responses made under different conditions.

• Sometimes the comparisons are *within* subjects:
  
  – E.g. blood pressure and temperature after exercise, compared with same before exercise, for the same subject.

• Sometimes *between* subjects:
  
  – E.g. blood pressure and temperature after exercise for a group of athletes, compared with same for a group of couch potatoes.
• Some problems will have both *within-subjects* factors and *between-subjects* factors.

  – E.g. blood pressure and temperature... (fill in as an exercise!).

• We begin with experiments involving only within-subjects factors...
Paired Comparisons

- Recall univariate paired comparisons:

  - $X_{j,i} =$ measurement on unit $j$, $1 \leq j \leq n$, for treatment $i$, $i = 1, 2$.

  - Analysis is based on $D_j = X_{j,1} - X_{j,2}$, through their summary statistics $\bar{D}$ and $s_d$.

  - The hypothesis $H_0 : \mathbb{E}(D) = \delta$ is tested using
    
    $$t = \frac{\bar{D} - \delta}{s_d/\sqrt{n}}$$
    
    and a confidence interval for $\delta$ is
    
    $$\bar{D} \pm t_{n-1}(\alpha/2)\frac{s_d}{\sqrt{n}}.$$
• Multivariate extension:

  – $X_{i,j,k}$ = measurement of variable $k$, $1 \leq k \leq p$, on unit $j$, $1 \leq j \leq n$, for treatment $i$, $i = 1, 2$.

  – The $p$ differences for each unit $D_{j,k} = X_{1,j,k} - X_{2,j,k}$ make a multivariate response $\mathbf{D}$ with summary statistics $\bar{\mathbf{D}}$ and $S_d$.

• The hypothesis $H_0 : \mathbf{E}(\mathbf{D}) = \delta$ is tested using

\[
T^2 = n \left( \bar{\mathbf{D}} - \delta \right)' S_d^{-1} \left( \bar{\mathbf{D}} - \delta \right)
\]
• Reducing the data to the differences $\mathbf{D}$ makes this problem the same as making inferences about the mean of a single sample:

  – The distribution of $T^2$ under $H_0$ is:
    \[
    \frac{(n - p)}{(n - 1)p} T^2 \sim F_{p, n-p},
    \]
    so we reject $H_0$ at level $\alpha$ if
    \[
    T^2 > \frac{(n - 1)p}{(n - p)} F_{p, n-p}(\alpha).
    \]
    – Confidence statements about $\delta$ are made as before.
• Example: wastewater treatment plant effluent:

  - \( n = 11 \) samples of wastewater;

  - \( p = 2 \) variables, Biochemical Oxygen Demand and Suspended Solids;

  - 2 “treatments” are two labs being compared, one Commercial and one State.

  - For each sample, a total of \( q = 4 \) measurements.

• SAS \texttt{proc glm} and \texttt{proc reg} program and output.
Comparing More Than Two Treatments

- Example: anesthetic test with dogs ($q = 4$ responses).
  - The 4 responses are the same physical measurement (milliseconds between heart beats) under 4 different experimental conditions:

<table>
<thead>
<tr>
<th>Treatment</th>
<th>CO₂</th>
<th>Halothane</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>high</td>
<td>without</td>
</tr>
<tr>
<td>2</td>
<td>low</td>
<td>without</td>
</tr>
<tr>
<td>3</td>
<td>high</td>
<td>with</td>
</tr>
<tr>
<td>4</td>
<td>low</td>
<td>with</td>
</tr>
</tbody>
</table>

- This is a repeated measures design.
• We need to test various hypotheses:

  – No difference among treatments: \( \mu_1 = \mu_2 = \mu_2 = \mu_4 \).

  – If we reject that null hypothesis, we will test main effects of \( \text{CO}_2 \) and Halothane, and their interaction.

• Each hypothesis can be written in the form

  \[ H_0 : C\mu = 0 \]

  for an appropriate \( q' \times q \) matrix \( C \).
• For example, the hypothesis $\mu_1 = \mu_2 = \mu_2 = \mu_4$ can be written

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with $q' = 3$.

• The main effect of Halothane, for example, is estimated by

$$(\mu_3 + \mu_4) - (\mu_1 + \mu_2) = [-1, -1, 1, 1] \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix}$$

with $q' = 1$. 
For each such hypothesis, the statistic
\[ T^2 = n(C\bar{x})'(CSC')^{-1}(C\bar{x}) \]
is the appropriate Hotelling's \( T^2 \) test statistic, and under \( H_0 \),
\[ \frac{(n - q')}{(n - 1)q'} T^2 \sim F_{q',n-q'}, \]

- SAS proc glm program and output.