Theoretical Background I

- Herglotz’s theorem: if \( \{\gamma_r\} \) is non-negative definite, then \( \exists \) non-decreasing \( S(f) \) such that

\[
\gamma_r = \int_0^1 e^{2\pi ifr} dS(f), \quad -\infty < r < \infty.
\]  

(1)

- Interestingly, the proof is closely related to the periodogram, although Herglotz was unaware of the connection.
• Define

\[ s_n(f) = \sum_{r=-(n-1)}^{n-1} \left( 1 - \frac{|r|}{n} \right) \gamma_r e^{-2\pi i fr}, \quad n \geq 1, \]

and

\[ S_n(f) = \int_0^f s_n(f') df'. \]

• Then \( \{S_n(\cdot)\} \) has convergent subsequences, and every limit \( S^*(\cdot) \) of a convergent subsequence satisfies

\[ \gamma_r = \int_0^1 e^{2\pi i fr} dS^*(f), \quad -\infty < r < \infty. \quad (2) \]
• But all $S^*(\cdot)$ satisfying (2) must be the same (at continuity points), so in fact

$$S_n(f) \to S(f), \quad n \to \infty$$

at all $f$ where $S(f)$ is continuous.

• The connection: $s_n(f)$ is just the expected value of the periodogram of $X_0, X_1, \ldots, X_{n-1},$

$$s_n(f) = \mathbb{E}[I_n(f)].$$
Lebesgue Decomposition

- In general, the spectral distribution function $S(\cdot)$ can be written as the sum of:

  - an absolutely continuous part $S_{ac}(\cdot)$ which is the indefinite integral of the corresponding spectral density function $s(\cdot)$:

    $$S_{ac}(f) = \int_0^f s(f')df';$$

  - a discrete part:

    $$S_d(f) = \sum_j R_j \mathbb{1}_{[f_j,1]}(f), \quad R_j > 0;$$
— a continuous but singular part, which we always assume to be absent.
• The contribution of the discrete part to $\gamma_r$ is

$$\sum_j R_j e^{2\pi i f_j r}$$

and because this must be real, the frequencies $f_j$ can be paired up and the sum rewritten as

$$\sum_{j: 0 \leq f_j \leq \frac{1}{2}} R_j^* \cos(2\pi f_j r)$$

where

$$R_j^* = \begin{cases} R_j & f_j = 0 \text{ or } \frac{1}{2} \\ 2R_j & \text{otherwise.} \end{cases}$$
• But this is just the autocovariance sequence of the series

\[ X_t = \sum_{j:0 \leq f_j \leq \frac{1}{2}} \left[ A_j \cos(2\pi f_j t) + B_j \sin(2\pi f_j t) \right], \]

which is weakly stationary provided:

\[ \mathbb{E}(A_j) = \mathbb{E}(B_j) = 0, \]
\[ \mathbb{E}(A_j^2) = \mathbb{E}(B_j^2) = R_j^* \]

and all \( A_s \) and \( B_s \) are uncorrelated.

• We have seen how to remove terms like this from an observed series, so we shall usually assume further that the discrete component is absent.
Spectral Density as a Fourier Series

• When \( S(\cdot) \) is absolutely continuous, (1) becomes

\[
\gamma_r = \int_0^1 e^{2\pi ifr} s(f) df, \quad -\infty < r < \infty.
\]

• That is, \( \gamma_r \) is the \( r^{th} \) Fourier coefficient of \( s(\cdot) \).

• Consequently, \( s(\cdot) \) may be “represented by” the Fourier series

\[
s(f) = \sum_{-\infty < r < \infty} \gamma_r e^{-2\pi ifr}.
\]
• The question of when such a series actually converges is delicate:

\[
s(f) = \lim_{n \to \infty} \sum_{r=-n}^{n} \gamma_r e^{-2\pi i f r}.
\]

(3)

• Simplest case: if \( \sum_{-\infty < r < \infty} |\gamma_r| < \infty \), then (3) converges uniformly \( \forall f \in [0, 1] \).

• But then \( s(\cdot) \) is necessarily bounded:

\[
0 \leq s(f) = \left| \sum_{-\infty < r < \infty} \gamma_r e^{-2\pi i f r} \right| \leq \sum_{-\infty < r < \infty} |\gamma_r|,
\]

and in fact, \( s(\cdot) \) is continuous, uniformly on \([0, 1]\).
• Some important models, including ARMA, satisfy this condition.

• We shall be interested in some other cases, including unbounded $s(\cdot)$, so we need some more general results.

• The Dirichlet-Dini theorem implies that \((3)\) converges at every $f$ where $s(\cdot)$ is differentiable.
• Because 
\[ \gamma_0 = \int_0^1 s(f) df, \]
the spectral density is always integrable.

• If it is, in addition, square-integrable, 
\[ \int_0^1 s(f)^2 df < \infty, \]
then (3) converges in mean square, 
\[ \lim_{n \to \infty} \int_0^1 \left[ s(f) - \sum_{r=-n}^{n} \gamma re^{-2\pi ifr} \right]^2 df = 0, \]
and in fact (Carleson) almost everywhere.