Black-Scholes Option Price

- The price at time t of a stock that pays no dividend is S.
- We want to price a derivative that pays $f(S_T)$ at time T.
- For example:
  - A call option with strike K has payoff
  $$f_{\text{Call}}(S_T) = (S_T - K)_+$$
  - A put option with strike K has payoff
  $$f_{\text{Put}}(S_T) = (K - S_T)_+$$

Assumptions

- Under the real probability measure $\mathbb{P}$, the stock price is a Geometric Brownian Motion:

  $$dS = \mu S dt + \sigma S dB$$

  where B is a standard Brownian Motion (Wiener process).

  - Here μ is the rate of return, and σ is the volatility.

- The economy also contains a risk-free asset that has a constant rate of return r.

No Arbitrage

- We assume no arbitrage exists, so a risk-neutral measure $\mathbb{Q}$ exists.

- Special property of Brownian Motions: because $\mathbb{Q}$ is equivalent to $\mathbb{P}$, B is still a Brownian Motion under $\mathbb{Q}$, and still with volatility 1; only the drift can change.

- In fact, under $\mathbb{Q}$, B has drift $(r - \mu)t$, so if we write $B^* = B - (r - \mu)t$, then

$$dS = rSdt + \sigma SdB^*$$

where B^* is a standard Brownian Motion under $\mathbb{Q}$.

Process for Log Price

- Rewrite:

$$\frac{dS}{S} = rdt + \sigma dB^*.$$

- Ito's Lemma:

$$d\log(S) = \frac{dS}{S} - \frac{\sigma^2}{2} = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dB^*.$$

- So for $t < T$, conditionally on $S_t = s_t$,

$$\log(S_T) - \log(s_t) \sim N\Big[(T-t)\left(r - \sigma^2/2\right), (T-t)\sigma^2\Big].$$

Derivative Price

- The price at time $t < T$ of the derivative with payoff $f(S_T)$ at time T is the expected value under $\mathbb{Q}$ of the payoff, discounted to time t, conditional on the price at time t being s_t:

$$e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[f(S_T)|\, S_t = s_t]\,.$$

- The conditional distribution of S_T is the lognormal distribution on the previous slide.

Call Option

- For the call option with payoff

$$f_{\text{Call}}\,(S_T) = (S_T - K)_+$$

some tedious calculus shows that the price at time t is

$$p_{\text{Call}}(s_t, T-t, K, \sigma, r) = s_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)\,.$$

- Here $\Phi(\cdot)$ is the normal cumulative distribution function,

$$d_1 = \frac{\log\left(\frac{s_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

and $d_2 = d_1 - \sigma\sqrt{T-t}$.

Implied Volatility

- Given $s_t, T - t, K, \sigma$, and r, we can compute the fair price of the option.

- Alternatively, and more commonly, given the price and $s_t, T - t, K$, and r, we can solve for σ: the implied volatility.

- Implied volatility is usually higher than historical volatility.

- At any given time, we may have prices for several options with different strikes and maturities; each corresponds to its own implied volatility.