

# 1 Long Term Risk

Long term risk is inherently credit risk, that is the risk that a counterparty will fail in some contractual obligation. Market risk is of course capable of causing damage over the long term, but no new issues arise: if market risk is adequately managed in the short term, the same or very similar strategies will contain it for the long term.

Market variability may also be a factor in credit risk. In the past, credit risk has been an issue principally for banks, as a result of making loans. The exposure on a loan is essentially the outstanding amount, and is not affected by market conditions. With derivatives, however, the level of exposure is driven by market conditions, and is largely unpredictable. This unpredictability substantially complicates the issue.

The principal new tool that can be brought to bear in studying long term risk is the quantification of *default risk*. In studying short term risk, credit risk is acknowledged but sometimes not quantified, and is managed by the use of collateral, among other approaches. On the time scale of years it is possible to assess the *probability* of default, and to bring this into the quantification of risk.

## 1.1 Default Risk

There is a great deal of information about the risk that an individual or an entity will default on an obligation. Information about an individual is summarized (incompletely, and apparently often incorrectly) in a *credit history*. Information about entities, at least those that issue significant amounts of debt, is collected by *credit rating agencies* such as Moody's Investors Service and Standard & Poor's Ratings Group.

### 1.1.1 Individual Defaults

On an individual level, standardized ways of scoring an individual's credit history now give a moderate degree of skill in assessing the probability that a mortgage-holder or credit card-holder will default on the corresponding obligations. These permit statements to be made about the behavior of *pools* of loans, which in turn allows such pools to be *securitized*:

- The owner of the loans forms a *trust*, a separate legal entity, to which it sells the loans, and which then sells certificates to investors.

- The volume of loans sold is large enough that there is a high probability that the pool will deliver:
  - enough interest payments to cover interest payments due to the holders of the certificates; and
  - enough return of principal to cover the redemption of the certificates at their maturity.
- The original lender may
  - retain the right to receive all amounts in excess of requirements, or
  - sell such rights as a *junior tranche* of certificates, subordinated to the *senior tranche*.

Certificates like these are referred to as:

**collateralized** for example Collateralized Mortgage Obligations (CMOs);  
or

**asset-backed** for example credit card receivables and car loan receivables.

In this way, the institution originating the loan both raises capital to make new loans and at least partly shields itself from the risk of default by its borrowers. Statistical analysis of credit histories and probabilistic modeling of the benefits of senior status allow the risks involved in these securities to be analyzed, and usually give the credit rating agencies enough comfort for them to receive high ratings (Aaa/AAA). A strong rating like this makes the securities acceptable even to conservative investors like pension funds.

### 1.1.2 Credit Ratings

When an institution issues *debt*, such as:

- corporate bonds, sold by a manufacturer to raise working capital;
- collateralized or asset-backed securities, issued by a financial institution to manage risk and raise capital for loans; and
- municipal bonds, issued by a government entity to raise funds for public projects such as school building, highways, or other facilities;

the soundness of the project is often evaluated by a *rating agency*. The evaluation is typically carried out in terms of the dependability of the cash flows that will be used to *service* the debt, and is based on appropriate characteristics of each type of issuer, such as:

- ratio of debt to equity, among other things, for a corporate issuer;
- quality and diversification of borrowers, for an asset-backed security;
- budgetary discipline and (since Orange County’s 1994 bankruptcy) taxpayer sentiment for a municipal issuer;

together with any special features such as assets (financial or physical) pledged to secure the debt. The evaluation allows the agency to assign a *rating* to the debt, on a scale such as

	Moody’s	Standard & Poor’s
	Aaa	AAA
Investment Grade:	Aa1, Aa2, Aa3 A1, A2, A3 Baa1, Baa2, Baa3	AA+, AA, AA- A+, A, A- BBB+, BBB, BBB-
Sub-Investment Grade:	Ba1, Ba2, Ba3 B1, B2, B3 Caa	BB+, BB, BB- B+, B, B- CCC

Ratings like these have been used for many years, and the agencies have followed them up to find how they change over time, and in particular the frequency with which entities of a given rating go into default. The frequency of rating transitions over a single year found by Standard & Poor’s, in percentages, is:

	AAA	AA	A	BBB	BB	B	CCC	D	NR
AAA	88.72	8.14	0.66	0.06	0.12	0.00	0.00	0.00	2.29
AA	0.68	88.31	7.59	0.62	0.06	0.14	0.02	0.00	2.58
A	0.09	2.19	87.74	5.32	0.71	0.25	0.01	0.06	3.64
BBB	0.02	0.31	5.61	81.95	5.00	1.10	0.11	0.18	5.72
BB	0.03	0.13	0.61	7.03	73.27	8.04	0.91	1.06	8.93
B	0.00	0.10	0.21	0.38	5.66	72.91	3.56	5.20	11.98
CCC	0.18	0.00	0.18	1.07	1.96	9.27	53.48	19.79	14.08

Each row gives the frequency of transitions from a given rating to all ratings. “D” means “In Default”, and is always treated as an absorbing state. “NR” means “Not Rated”. There are two reasons why an entity may become “Not Rated”:

- its level of outstanding debt falls below \$25 million;
- it requests the rating to be withdrawn.

The former is presumably a sign of financial health; the latter often precedes a default. Standard & Poor’s does not publish information about transitions out of “NR”, but the possibility of subsequent default obviously cannot be ignored.

Standard & Poor’s does publish some information about multi-year transitions, in particular *cumulative default frequencies*: the frequency with which an entity of a given initial rating is found to be in default at or prior to a given number of years later. These are shown below:

	1 yr	2 yr	3 yr	4 yr	5 yr	6 yr	7 yr	8 yr	9 yr	10 yr
AAA	0.00	0.00	0.07	0.15	0.24	0.43	0.66	1.05	1.21	1.40
AA	0.00	0.02	0.12	0.25	0.43	0.66	0.89	1.06	1.17	1.29
A	0.06	0.16	0.27	0.44	0.67	0.88	1.12	1.42	1.77	2.17
BBB	0.18	0.44	0.72	1.27	1.78	2.38	2.99	3.52	3.94	4.34
BB	1.06	3.48	6.12	8.68	10.97	13.24	14.46	15.65	16.81	17.73
B	5.20	11.00	15.95	19.40	21.88	23.63	25.14	26.57	27.74	29.02
CCC	19.79	26.92	31.63	35.97	40.15	41.61	42.64	43.07	44.20	45.10

The published data in fact go out to 15 years, but for some initial ratings there is no further change, apparently reflecting sparsity of data.

It is plausible to view the rating changes of an entity as a *Markov chain*. The current rating is intended to summarize the status of the entity, and its prior rating history should be irrelevant to the probability that it will be in any specified category at a given future time (or at a minimum, to the probability that it will be in default). This is the substance of the Markov assumption.

Under the Markov assumption, the probability of making a transition between two specific states over a specified period of time, a *multi-year* transition probability, can be computed from the single-year transition probabilities. However, the probabilities of transitions among all states must be given.

That is, the matrix must be completed with rows for initial states “D” and “NR”. The transition probabilities from “D” are determined by its treatment as an absorbing state: with probability 1, an entity in “D” remains in “D”.

Probabilities for “NR” may be inferred from the cumulative default rates. The row

	AAA	AA	A	BBB	BB	B	CCC	D	NR
NR	90.74	0.00	0.00	0.00	0.00	0.00	8.73	0.53	0.00

gives a reasonable approximation to the multi-year cumulative default rates (in the sense of total squared Hellinger distance):

	1 yr	2 yr	3 yr	4 yr	5 yr	6 yr	7 yr	8 yr	9 yr	10 yr
AAA	0.00	0.01	0.07	0.16	0.26	0.38	0.51	0.66	0.82	0.99
AA	0.00	0.03	0.12	0.26	0.42	0.61	0.82	1.05	1.30	1.57
A	0.06	0.16	0.36	0.62	0.92	1.27	1.64	2.04	2.45	2.87
BBB	0.18	0.49	0.99	1.60	2.28	3.00	3.74	4.47	5.19	5.89
BB	1.06	2.50	4.20	5.95	7.66	9.24	10.69	11.98	13.14	14.17
B	5.20	9.82	13.88	17.29	20.10	22.40	24.27	25.79	27.04	28.07
CCC	19.79	30.95	37.63	41.75	44.40	46.18	47.43	48.35	49.05	49.60

The “inferred” transition probabilities out of “NR” are not offered as being especially plausible for “NR” entities. In fact, as noted above, this is likely to be a nonhomogeneous category, and it is consequently unlikely that the Markov assumption holds. However, the inferred row is useful for other calculations where a complete matrix is needed.

## 1.2 Pricing Credit Risk—The Bond Market

The credit risk in a bond can vary from essentially none, in U.S. Treasury obligations, to quite high in “high yield” (junk) bonds. Consider an investor who has to choose between investing \$100 in either:

- a Treasury note maturing in 5 years, with a coupon of 6.5% per annum; or
- a bond issued by a BBB (that is, low investment grade) issuer paying the same coupon.

The 6.5% coupon on the Treasury note is the current yield, so the note is priced “at par”: its current price is \$100 for a note with a face value of \$100. What is the appropriate price for the BBB-rated bond?

If attention is focused on the return of principal after 5 years, an investment in the Treasury note returns \$100 with probability 1, while the issuer of the BBB bond has a 1.78% chance of having defaulted. Thus the return to the investor is

$$\begin{aligned} & \$100 \quad \text{with probability } .9822 \\ & \$100R \quad \text{with probability } .0178 \end{aligned}$$

where  $R$  is the random *recovery* on a defaulted bond. The expected value of the return is thus  $\$100[1 - .0178E(1 - R)]$ . It is reasonable to take  $E(R) = .5$ , giving an expected return of \$99.11, or an expected loss of \$0.89.

If the investor were willing to pay a price that exactly compensated for this expected loss, this price would be \$0.89 less than that of the Treasury note, namely \$99.11. The approximation

$$\frac{\delta \text{price}}{\text{price}} = \text{duration times } \delta \text{ yield}$$

combined with a duration of 4.35 years shows that  $\delta \text{yield}$  should be  $0.89/4.36 = 0.20\% = 20 \text{ basis points}$  (a basis point is 0.01%). That is, the BBB bond would be priced to yield  $6.50 + 0.20 = 6.70\%$ ; it would *trade* at 20 basis points higher than the Treasury note, or at a *spread* of +20 basis points. The spread is the additional interest that the investor should be paid to compensate for carrying the default risk in the issuer.

A similar calculation for other maturities gives the following *term structure* for BBB credit spreads:

1 yr	2 yr	3 yr	4 yr	5 yr	6 yr	7 yr	8 yr	9 yr	10 yr
9	12	13	18	20	24	26	28	28	29

In fact, 5-year BBB debt tends to trade at around +80 basis points, because investors demand to be compensated for more than expected loss. Part of this spread is attributable to tax treatment of interest payments—perhaps 25 basis points. The balance, around 60 basis points, is therefore consistent with a *risk-adjusted* default probability that is around three times the historical default probability.

### 1.2.1 Credit Spreads

To see why the credit spread is so large, consider the problem faced by the investor. Suppose that the investor can *diversify* the holdings over several issuers, and for the sake of simplicity suppose that all are rated BBB and all obligations mature in 5 years. The investor might be the manager of a pension fund, and owes an obligation to its depositors to have the entire investment available at that time. To be reasonably sure that it will have sufficient funds, it must set aside a sum of *capital* to cover the losses.

Let  $P$  be the total to be invested, and write  $p$  for the probability of default of each of the  $N$  issuers. Suppose for the moment that issuers default independently. For simplicity again, suppose that the recovery rate  $R$  is 0.5 with probability 1. Then the amount lost to defaults is distributed as  $(P/2N)X$ , where  $X$  is binomially distributed  $B(N, p)$ . Since  $p$  is small this is approximately Poisson with mean  $Np$ , whence  $\sqrt{4X}$  is approximately  $N(\sqrt{4Np}, 1)$ . This implies that with probability  $1 - \alpha$  the loss will not exceed

$$\frac{P}{8N} \left( \sqrt{4Np} + Z_\alpha \right)^2$$

where  $Z_\alpha$  is the upper  $\alpha$  quantile of the standard normal distribution ( $Z_{.05} = 1.645$ ,  $Z_{.01} = 2.326$ , for example). This is the amount of capital required to cover losses with probability  $1 - \alpha$ .

Of this amount, the expected loss of  $Pp/2$  would be designated as *reserves*. The balance,

$$\frac{P}{8N} \left( 2Z_\alpha \sqrt{4Np} + Z_\alpha^2 \right)$$

is charged against the institution's *equity*, that is, the interests of its various owners. If its owners require it to earn  $r\%$  on its equity (return on equity, or ROE), it must generate

$$\left( \frac{r}{100} \right) \frac{P}{8N} \left( 2Z_\alpha \sqrt{4Np} + Z_\alpha^2 \right)$$

in income, *beyond* the expected loss of  $Pp/2$ .

To be specific, if  $N = 20$  and  $\alpha = .01$ , then the equity is 6.85% of  $P$ . If  $r = 16.5\%$ , and the capital is invested safely in Treasurys earning 6.5%, the additional income is 10% of this amount, or 0.685% of  $P$ . This set of requirements would lead therefore to a spread of around 70 basis points in addition to the 20 basis points required to cover expected loss, leading to an overall spread of 90 basis points, close to what is observed in the market.

### 1.2.2 Constraints

Various constraints are evident in these calculations. First, one could in principal make the the equity requirement arbitrarily small by taking  $N$  to be sufficiently large. There are indeed a large number of BBB (and similar) issuers. However, defaults were assumed earlier to be *independent* random events, and concentrations of issuers in similar sectors and markets means that only a limited number could be regarded as stochastically independent. More careful analysis might use a model for correlated defaults; to a good approximation, it is enough to view a large number of nonindependent risks as being equivalent to a smaller number, perhaps 20, of independent risks.

It is also clear that if  $r$  is large and  $\alpha$  is small, the required spread may exceed what is available in the market. Thus a high ROE is incompatible with a low  $\alpha$ . The example of  $\alpha = .01$  or 1% fits between the BBB and A 5-year default probabilities, and is not a very high standard. For the manager to achieve a AAA rating for its management, it would have to be more stringent (around 0.2%, or  $\alpha = .002$ , with  $Z_{.002} = 2.88$ ), and accept a correspondingly lower ROE.

### 1.3 Credit Risk in Derivatives

The long term credit risk in a portfolio of derivatives may be approached in a way that parallels the previous discussion of the bond market. The major difference is that whereas the amount at risk in owning a bond is essentially always equal to the face value of the bond, in a derivative transaction the amount at risk, the *replacement cost* of the position, varies with market conditions. This requires that we model the default process and market processes simultaneously.

Because a portfolio of derivatives may contain many different types of instrument of different maturities, there may be different degrees of risk at different horizons. Write  $L(t)$  for the aggregate credit loss experienced up to time  $t$ . Clearly most questions about the magnitude of the risk can be answered using the probability distributions of  $L(t)$  for various values of  $t$ , say  $t_1, t_2, \dots, t_n$ . For a complete description one might want the *joint* distribution of  $L(t_1), L(t_2), \dots, L(t_n)$ . However, the most useful information is contained in the marginal distributions.

Suppose that the institution of interest has derivatives positions with  $N$  counterparties, and that the exposure to the  $i$ th counterparty at time  $t$  and

under market conditions  $m$  is  $V_i(t, m)$ . Suppose that the time to default of this counterparty is  $T_i$  and that the recovery factor in the default is  $R_i$ . Then

$$L(t) = \sum_{i:T_i \leq t} (1 - R_i)V_i[T_i, m(T_i)] = \sum_i (1 - R_i)V_i[T_i, m(T_i)]H(t - T_i)$$

where  $H(x)$  is the step function,  $H(x) = 1, x \geq 0$ ,  $H(x) = 0, x < 0$ . Note that counterparty  $i$  causes no loss by time  $t$  if

- the default time  $T_i$  is longer than  $t$ , or
- the recovery factor  $R_i$  is 1, or
- all deals have matured before the default time  $T_i$ .

Thus the problem is to obtain the distribution of this sum.

### 1.3.1 Strategies for Computation

It is reasonable to assume that, conditionally on the values of appropriate stressors, the defaults of different counterparties are stochastically independent events, or equivalently that the  $T_i$ s are independent random variables. It is also reasonable to assume that, conditionally on the values of the same stressors, the recovery factors  $R_i$  are independent of  $T_i$  and of each other, and that the distribution of each depends in a known way on the characteristics of the counterparty. However, each term in the sum defining  $L(t)$  also involves  $m(T_i)$ , the market conditions at the time of default of counterparty  $i$ , and these cannot be assumed to be independent. Thus  $L(t)$  is a sum of dependent terms, and its distribution cannot be found from the (marginal) distributions of the summands. Various approaches may be adopted to find this distribution.

**Simulation** The need for simulation of market variability was mentioned in the context of short term risk (market risk). Similar approaches may be taken to construct simulated market histories out to the longer horizons of interest here. Some issues that are unimportant on the scale of days to weeks become relevant on a time scale of years, however. Principal among these is the *mean reversion* of interest rates. Although it may not be obvious from records of only a few years in extent, interest rates do not wander arbitrarily far from “typical” levels, even over periods of many years.

If an acceptable way can be found to simulate long term market histories and the stressors, it is then easy to simulate the default times independently, given say the cumulative default rates published by rating agencies and the initial rating of each counterparty.

Each simulation provides one realized value of  $L(t)$ . Since interest is likely to focus on the tail percentiles such as 1%, 0.2% or less, many simulations are needed to obtain accurate approximations. *Importance sampling* is essential to obtaining more accuracy with a given number of simulation runs.

**Convolution** Although the summands are not independent, because of the dependence on market conditions, they are *conditionally* independent, given the market path  $\{m(t'), 0 \leq t' \leq t\}$  and the corresponding path of the stressors. Thus the distribution of  $L(t)$  can be found in three conceptual steps:

- For each  $i$  and for a given market trajectory  $m(\tau), 0 \leq \tau \leq t$  and stressor trajectory, find the distribution of

$$L_i(t) = (1 - R_i)V_i[T_i, m(T_i)]H(t - T_i).$$

- Find the distribution of  $L(t) = \sum L_i(t)$  conditionally on this trajectory as the convolution of these distributions.
- Average the conditional distribution across market trajectories.

In this process, the first two steps can be carried out exactly; only the last needs to be based on simulation.

The computational procedure is closely related to these conceptual steps, and consists essentially of the following:

- Simulate a single market trajectory  $m(\tau), 0 \leq \tau \leq t$  and stressor trajectory at the desired finite set of times.
- For each  $i$  and for the given trajectory, find the distribution function  $F_i(l|m; t) = P[L(t) \leq l|m]$  of

$$L_i(t) = (1 - R_i)V_i[T_i, m(T_i)]H(t - T_i).$$

- Find the distribution function  $F(l|m; t) = P[L(t) \leq l|m]$  of  $L(t) = \sum L_i(t)$  conditionally on this trajectory as the convolution  $F_1(\cdot|m; t) * F_2(\cdot|m; t) * \dots * F_N(\cdot|m; t)$  of these distributions.
- Average the conditional distribution across simulations.

### 1.3.2 Affiliate Risk and Sovereign Risk

**Affiliates** It is clear that credit exposures to affiliated entities cannot be treated as if the default times are independent. In fact, it is a reasonable approximation to treat them as *identical*. The situation is therefore the same as if we were facing a single super-counterparty, to which our exposure is the sum of the exposures to the affiliates.

One complicating factor is that the legal contracts comprising the deals are still with the individual affiliates, and it is not safe to assume that obligations to different affiliates may be set off against each other. Thus the exposure to the group must be treated as the sum of the positive exposures, neglecting any negative values:

$$V_{\text{group}} = \sum_i \max(V_i, 0).$$

**Sovereign Risk** The possible actions of a foreign government also change the structure of the risks. Governments have the power to control payments across their borders, and could for instance prohibit certain kinds of payment. If we have counterparties in a country whose government introduces such controls, we may lose any assets in the corresponding deals.

It is claimed that the probabilities of such events fit into the same structure as defaults by issuers of debt, and indeed credit ratings have been announced for all countries that are active in the derivatives market. The possibility that the action of *one* entity, namely the government, can cause losses in deals with *several* counterparties again introduces stochastic dependence into the times at which such losses occur.

Losses must now be grouped by country. Losses to different countries are still independent (after conditioning on the market path, if path-dependent default rates are being used), whence convolution may still be used to find the overall distribution of losses. Within a single country, times to loss are dependent, but become independent again when the calculation is conditioned on the time of the sovereign event. This allows the distribution of losses aggregated over a single country to be computed exactly, with a little added complication.

Affiliates that fall under the jurisdiction of different sovereigns raise complex questions, legal as well as probabilistic. The simplest treatment is to view the parent as the responsible party, and ignore the possibility of loss

caused by the actions of the sovereign of a jurisdiction other than that of the parent.

## 1.4 Implications of Credit Risk

We have seen how to measure long term credit risk in terms of the probability distribution of  $L(t)$ , the cumulative credit losses we may face up to a horizon of  $t$  years in the future. We can use this distribution to manage the risk.

### 1.4.1 Expected Losses

The amount we may *expect* to lose over this horizon is  $E[L(t)]$ . It is usual to set up *reserves* in this amount, which consist of revenues that are not recognized immediately but rather *deferred* to cover any losses that may actually occur. Since credit losses are aggregated first over deals with a given counterparty and then over counterparties, the overall expected loss may in turn be disaggregated to the counterparty level and if necessary to the deal level.

### 1.4.2 Unexpected Losses

There is of course a significant probability that losses will exceed their expected value. Because of skewing of the distribution, it is typically less than 50%, but still high enough to be of concern. A prudent business would want to make provision to absorb losses to a much higher probability, say 99% over a single year. This is the role of *capital*, which is the sum of reserves and *equity*.

For some closely observed businesses, the required level of capital is in fact determined in part by a calculation of the distribution of credit losses. In order to achieve a high (triple-‘A’) credit rating, an entity may have to persuade the rating agency that it has a very high probability of survival for a lengthy period. The required capital may be set simply at a high percentile, for instance so that for all horizons  $t$ ,

$$pr[L(t) > \text{capital}] < pr[\text{default of a triple-‘A’ bond of maturity } t]$$

Alternatively it may be set in terms of the amount that one of its counterparties may expect to lose:

$$E(\text{loss}) = pr(\text{default}) \cdot E(1 - \text{recovery} | \text{default})$$

which involves

$$E[\max(L(t) - \text{capital}, 0)]$$

as well as estimates of the magnitudes of assets and liabilities.