

1 Marking to Market

An essential first step in managing the risk in a portfolio of financial instruments is setting up the capability of *valuing* the individual instruments. The value of any instrument is usually defined as its *replacement cost*. For an asset like a bond, that is simply its market price. For a derivative such as a swap, it is the price you would have to pay to enter the transaction (or what remains of it) on the valuation date.

For a small portfolio of simple (“plain vanilla”) instruments, you could simply call a sample of dealers and obtain quotes for the replacement cost. For a larger portfolio or one with more complex deals, this would quickly become infeasible. Standard practice is to use a *model*, calibrated to the prices of a manageable number of plain vanilla products.

1.1 General Approach

A financial instrument may be viewed as consisting of a set of *cash flows* occurring on specified *payment dates*. Write t_i for the i 'th payment date, and C_i as the corresponding cash flow. Use the conventions that the time origin $t = 0$ is the valuation date, and that $C_i > 0$ represents a payment to be *received* by the entity carrying out the valuation.

The magnitude of a future cash flow may or may not be known at the time of valuation. An unknown cash flow is modeled as a random variable, whose (risk-neutral) distribution is conditioned on all information available at the time of valuation. The probability distributions of such random variables are discussed below.

In valuing the deal, it is assumed that instruments exist that could be used to replicate each cash flow in isolation. If C_i is a fixed payment, the instrument is a *zero-coupon bond* maturing at time t_i . If it is of unknown magnitude, for instance the 6-month LIBOR rate 6 months into the future, the instrument is a futures contract.

Valuation models are based on the following principles.

- The replacement cost of a transaction is the sum of the values of the instruments replicating its individual cash flows. If this were not so, an enterprising *arbitrageur* could make an instant profit out of the difference.

- The value of a single cash flow C at a future time t is its expected value, discounted to present value: $E[C \times D(t)]$.
- Often the discount factor $D(t)$ is nonrandom, and then the present value can be written $E(C) \times D(t)$.

The discount factor may be written as $D(t) = e^{-tr(t)}$, where $r(t)$ is the interest rate (yield) for risk-free investments maturing at time t . The discount factor $D(t)$ is also the market price of a zero-coupon bond returning \$1 at time t .

Thus the value of the transaction is

$$\sum_i D(t_i)E(C_i).$$

1.2 Examples

zero-coupon bond: A zero-coupon bond returns its face amount, say \$1, on its maturity date T . There is just one cash flow, hence the sum has just one term, with value $D(T)$: the value of a zero-coupon bond is just the discounted value of its face amount. Treasury bills of 1 year maturity or less pay no coupons, and the Treasury also sells zero-coupon bonds of maturities up to 30 years; the market in these instruments can be used to determine discount factors. LIBOR rates are also for deposits that make no recurring interest payments, and are typically used instead of Treasury rates, because they are more closely related to the borrowing costs of financial institutions.

coupon-bearing bond: A bond with principal amount P , a maturity of T years, and with an annual interest rate or *coupon* $C\%$ has cash flows $\$PC/2$ twice a year prior to maturity, and $\$P + PC/2$ at maturity. Its value is therefore

$$V = \sum_{i=1}^{2T-1} P \frac{C}{2} D(i/2) + \left(P + P \frac{C}{2} \right) D(T). \quad (1)$$

Its *yield to maturity* is defined as the *single* yield r that gives the same value; that is, the solution to

$$V = \sum_{i=1}^{2T-1} P \frac{C}{2} e^{-ir/2} + \left(P + P \frac{C}{2} \right) e^{-rT}. \quad (2)$$

It is the yields to maturity of bonds that are usually reported in the news. For instance the yield on a 30-year Treasury bond is currently 4.38%;¹ this is its yield to maturity.

floating rate note: A floating rate note is like a bond, with recurring interest payments, but the interest rate floats with the market instead of being fixed for the life of the note. For example, with the interest rate reset quarterly, the rate for the first quarter is set at the issue date (actually 2 days prior), and determines the interest payment at the end of that quarter. The rate for the second quarter is set at (again, actually 2 days prior to) the date of the first interest payment, and so on. The rate is usually tied to LIBOR, often with a *spread*.

Since the interest payments after the first are unknown at inception, they must be treated as random quantities, and in general we must make assumptions about their distribution to obtain their expected values. There is one special case where this is not needed: *if the floating rate is the corresponding zero-coupon yield, the value of the note is par at every reset date*. This is clear for the *last* reset date, since at this point the note is equivalent to a zero-coupon bond. The result follows by induction: if it is known in advance that the value at any given reset date will be par, we know that we could sell the note on that date for its face amount. Thus at the previous reset date it must have the same value as a zero-coupon bond maturing on the given reset date, which is again par.

interest rate swap: The parties to an interest rate swap exchange fixed interest rate payments for floating rate payments on a *notional* principal amount. If they were also to exchange the actual principal payments at the start and end dates (which would of course exactly offset each other), they would have replicated the cash flows of a fixed coupon bond and a floating rate note, respectively. Thus the value of the swap is that of a fixed coupon bond less that of a floating rate note, and in particular on each reset date is the difference between the current market price of the bond and par.

The zero-coupon yield $r(t)$ generally varies with t ; this dependence is called the *term structure* of yields. The graph of $r(t)$ against t is called the

¹<http://www.bloomberg.com/markets/rates/index.html>

zero-coupon *yield curve*. Yield curves for coupon-bearing Treasury securities such as those compiled by Bloomberg are more familiar; either can be obtained from the other.

Note that if yields change by a given amount at all maturities (a *shift* of the yield curve), the value of a long-term bond changes by more than that of a short-term bond, and the value of a zero-coupon bond changes by more than that of a coupon-bearing bond of the same maturity. A floating rate note has the same sensitivity as a zero-coupon bond with maturity equal to the *reset interval*, and is often regarded as essentially immune to interest rate risk. Actual changes in yields are never simple shifts (there are also *tilts* and other changes in shape), but shifts contribute much of the variability.

1.3 Duration

Interest rates or yields for maturities of longer than one year are essentially determined by bond prices. Since the price of a specific bond depends on its coupon as well as its maturity, it is not especially meaningful to construct an index of bond prices; yields are more general, and still allow the calculation of the price of any given bond.

It is qualitatively clear from (1) that longer-term bonds have prices that are more sensitive to interest rate fluctuations than those of shorter-term bonds. Equation (2) yields an interesting relationship:

$$-\frac{dV}{dr} = \sum_{i=1}^{2T-1} P \frac{C}{2} t_i e^{-rt_i} + \left(P + P \frac{C}{2} \right) T e^{-rT}$$

whence

$$\begin{aligned} \frac{d \log V}{dr} &= \sum_{i=1}^{2T-1} t_i \frac{P \frac{C}{2} e^{-rt_i}}{V} + T \frac{\left(P + P \frac{C}{2} \right) e^{-rT}}{V} \\ &= \sum_{i=1}^{2T-1} t_i w(t_i) + T w(T) \end{aligned}$$

where the $w(\cdot)$ s sum to 1. This is just a weighted average of the times of the cash flows, with weights proportional to the present values of the cash flows (discounted with respect to the single yield r), and is known as the *duration*

of the bond. Thus a small change in yields (strictly, in yield to maturity) of δr leads to a small change in value of δV , where

$$\frac{\delta V}{V} \approx -\text{duration} \times \delta r.$$

Note in particular that the duration of a zero-coupon bond is always equal to its maturity, but that the duration of a coupon-bearing bond is always less than its maturity. For instance the 30-year Treasury bond (maturing in August, 2039) carries a $4\frac{1}{2}\%$ coupon, and at a price of $\$102 + \frac{16.5}{32}$ gives a yield to maturity of 4.35% and a duration of 13.9 years.

1.4 Risk-Adjusted Probabilities

The probability distribution with respect to which the expected value $E(C)$ is calculated is not obvious. It is called a *risk-neutral* distribution, and it differs systematically from the “true” distribution (whatever that may be). Its existence is assured by theory, and it is inferred from the market prices of a sufficient number of instruments.

Risk-adjusted probabilities may be described loosely as the probabilities that would have to govern a world in which all participants were risk-neutral but that exhibited the currently observed real-world prices.

1.5 Option Pricing and Volatility

A derivative will often involve cash flows that are nonlinear functions of some underlying market variable. The simplest example is an option. For instance, a company with floating rate payment obligations may wish to avoid the risk of rising interest rates without taking on the downside risk of an interest rate swap. One solution is to buy a *cap*: the cash flow to the company on payment date t_i is

$$\text{Notional Principal} \times [L(t_i) - K]_+.$$

Here $L(t)$ denotes the floating rate, such as 6-month LIBOR, and K is the *strike* for the cap. The notation A_+ signifies the positive part of A , $\max(A, 0)$. The cash flow at time t_i is the payoff of a *call option* on $L(t_i)$ with strike K . Since the company only receives payments in the future, it must pay an up-front *premium* to buy the right to receive them.

The cap may be compared with the interest rate swap, in which the payment at time t_i may be written simply as

$$\text{Notional Principal} \times [L(t_i) - F].$$

Here F is the fixed rate, which is usually chosen to make the replacement cost zero at inception.

To value a cap, you need to find quantities like $E\{[L(t) - K]_+\}$, which depend on the entire distribution of $L(t)$. If the underlying, here $L(t)$, is lognormally distributed, the answer is the celebrated Black-Scholes option pricing formula. Under this simplistic assumption, the answer depends only the first two moments $E\{\log[L(t)]\}$ and $\text{Var}\{\log[L(t)]\}$. The latter is usually expressed in terms of the *volatility* $\sigma(t)$ in the form $\text{Var}\{\log[L(t)]\} = t\sigma(t)^2$. The time t is typically measured in years, and then the volatility is said to be *annualized*.

The volatility is constant if $\log[L(t)]$ follows a random walk, or *Wiener process*. Otherwise it exhibits a *term structure*. Interest rates are usually assumed to be stationary (mean-reverting), which implies that $\text{Var}\{\log[L(t)]\}$ is bounded and hence that $\sigma(t)$ must ultimately decline to 0.

1.6 Variation Over Time

All of the quantities discussed above are required to value a portfolio of derivatives *on a given day*. When the same portfolio is valued on another day, say one day later, all or most of them will have changed. The time variation, say of an interest rate or of a foreign-exchange rate, can be expected to be related to the parameters of the corresponding risk-neutral probability distribution, such as its volatility. However, *neither one can be inferred from the other*. In particular, there is a systematic difference between the *historical* and *implied* volatilities of any market variable.

Studying the statistical variation of market variables over time is very different from studying the risk-neutral probabilities implied by a given day's prices.