

1 Financial Risk

Financial risk is the *potential* for financial loss. All financial institutions face such risk, to some extent. Some events may lead to small losses, others to catastrophic losses. An institution's risk profile is characterized by the magnitudes of its potential losses, and their respective probabilities.

However, two different kinds of probabilities are used in financial mathematics; we need to understand how they differ and which is appropriate for risk characterization.

1.1 Actual Probabilities

Actual probabilities are derived from historical data, often by fitting a statistical model. For instance, a 2003 report by Standard & Poor's includes the information that on average from 1981 to 2002, 1.38% of firms that were rated BB on January 1 of a given year were in default by the end of the same year. If we assumed that each firm had the same probability of default in each year, we could use 0.0138 as an estimate of that probability. If we further assumed that events of default were statistically independent, we could assess the precision of that estimate. Neither of these assumptions is supported by the data, however—more on this later.

1.2 Risk-neutral Probabilities

Risk-neutral probabilities are derived quite differently, based on the market prices of various financial instruments. An outline of their development, summarized from Chapter 1 of Hunt and Kennedy, follows.

1.2.1 A single-period economy

Consider an economy \mathcal{E} containing n assets $A^{(i)}$, $i = 1, 2, \dots, n$. The values of these assets at the current time $t = 0$ are $A_0^{(i)}$, and we are concerned with only one future time $t = 1$. The economy has m possible states at $t = 1$, $\{\omega_j, j = 1, 2, \dots, m\}$, comprising the set Ω . The value of asset $A^{(i)}$ at $t = 1$ in state ω_j is $A_1^{(i)}(\omega_j)$.

We use the term *asset* in its usual sense of something that we want to own, so these values are all non-negative.

1.2.2 Arbitrage

To produce any theory, we have to assume that the economy has no opportunities for *arbitrage*. Arbitrage is making money with no risk. A *portfolio* is a collection of positions in the assets of the economy, and it is represented by a vector ϕ of *portfolio weights*. If we write A_0 and $A_1(\omega_j)$ as the column vectors of asset values at $t = 0$ and $t = 1$ respectively, then the values of the portfolio at those times are $\phi^T A_0$ and $\phi^T A_1(\omega_j)$.

The portfolio weight $\phi^{(i)}$ may be any real number, positive or negative. In a real economy, owning a fraction of a unit of some asset may not be feasible, so this assumption may be unrealistic. A negative weight represents a *short* position, in which the portfolio gains value if the asset loses value. A short position may be set up in a futures contract simply by selling the contract; a short position in stocks or bonds is usually set up by borrowing the asset and selling it, with the obligation to buy it later for return to the owner. Shorting Treasury instruments is usually feasible for a financial institution; shorting corporate bonds may be more difficult. By contrast, a positive weight represents a *long* position, which can be set up simply by purchasing the asset.

The economy \mathcal{E} admits arbitrage if there exists a portfolio ϕ satisfying either of the conditions

1. $\phi^T A_0 < 0$ and $\phi^T A_1(\omega_j) \geq 0$ for all j ;
2. $\phi^T A_0 \leq 0$ and $\phi^T A_1(\omega_j) \geq 0$ for all j , with strict inequality for some j .

In the first case, the revenue at time $t = 0$ from setting up the short positions is greater than the investment in the long positions, so setting up the portfolio yields cash, but in all possible states at time $t = 1$ the portfolio requires no cash to close it out; the investor has made a riskless profit.

In the second case, setting up the portfolio may not generate cash, but it does not require an investment, and yet at time $t = 1$ it either generates cash or at worst can be closed out at no cost.

1.2.3 Pricing kernel

A *pricing kernel* is any strictly positive vector Z with the property that

$$A_0 = \sum_j Z_j A_1(\omega_j).$$

Pricing kernels play a large role in asset pricing, as they allow the calculation of current prices from information about the value of the asset in all possible future states of the economy. The existence of a pricing kernel characterizes arbitrage-free economies.

Theorem: The economy \mathcal{E} is arbitrage-free if and only if there exists a pricing kernel Z .

1.2.4 Completeness

Suppose we wish to price some new instrument X with values $X_1(\omega_j)$ at time $t = 1$. We say that X can be *replicated* if there exists a portfolio ϕ with the same values: $\phi^T A_1(\omega_j) = X_1(\omega_j)$ for each j . Then the price of X at time $t = 0$ must be $\phi^T A_0$, because otherwise an arbitrage opportunity would exist. The economy is said to be *complete* if every possible instrument has a replicating portfolio.

Theorem: The economy \mathcal{E} is complete if and only if the matrix $A_1 = \left(A_1^{(i)}(\omega_j) \right)$ has full column rank, or in other words $A_1 x = 0$ implies that $x = 0$.

1.2.5 Uniqueness of the pricing kernel

We could also use a pricing kernel to price X :

$$X_0 = \sum_j Z_j X_1(\omega_j),$$

but this would not give a unique value unless the pricing kernel is itself unique. Not surprisingly, this is equivalent to completeness.

Theorem: Suppose the economy \mathcal{E} is arbitrage-free. Then it is complete if and only if there exists a unique pricing kernel Z .

Since existence of a pricing kernel implies absence of arbitrage, we can restate this as: there exists a unique pricing kernel Z if and only if the economy \mathcal{E} is arbitrage-free and complete.

1.2.6 Probabilities

Suppose that \mathbb{P} is a probability measure on Ω with $\mathbb{P}(\omega_j) > 0$ for all j .

Theorem: The economy \mathcal{E} is arbitrage-free if and only if there exists a strictly positive random variable Z such that

$$A_0 = \mathbb{E}[ZA_1].$$

Suppose further that for some asset $A^{(i)}$, $A_1^{(i)}(\omega_j) > 0$ for every j . Then the economy \mathcal{E} is arbitrage-free if and only if there exists a strictly positive random variable κ with $\mathbb{E}[\kappa] = 1$ such that

$$\mathbb{E}\left[\kappa \frac{A_1}{A_1^{(i)}}\right] = \frac{A_0}{A_0^{(i)}}.$$

The proof follows from constructing Z as $Z(\omega_j) = Z_j/\mathbb{P}(\omega_j)$ and κ as

$$\kappa(\omega_j) = Z(\omega_j) \frac{A_1^{(i)}(\omega_j)}{A_0^{(i)}}.$$

Theorem: Suppose that for some asset $A^{(i)}$, $\mathbb{P}(A_1^{(i)} > 0) = 1$ and $A_0^{(i)} > 0$. Then the economy \mathcal{E} is arbitrage-free if and only if there exists a probability measure \mathbb{Q} equivalent to \mathbb{P} such that

$$\mathbb{E}_{\mathbb{Q}}[A_1/A_1^{(i)}] = A_0/A_0^{(i)}.$$

With respect to this probability measure \mathbb{Q} , the quotients $A_t/A_t^{(i)}$ form a *martingale*, and \mathbb{Q} is called the *equivalent martingale measure* for $A^{(i)}$. Note that using a different asset as the denominator leads to a different equivalent martingale measure.

We can rewrite the last equation as

$$A_0 = \mathbb{E}_{\mathbb{Q}}\left[A_1 \frac{A_0^{(i)}}{A_1^{(i)}}\right]$$

or in words, prices at $t = 0$ are expected values (with respect to \mathbb{Q}) of prices at $t = 1$ *discounted to present value* by the *discount factor*

$$D = \frac{A_0^{(i)}}{A_1^{(i)}}.$$

These are the values that a risk-neutral investor would assign to the assets if that investor believed that \mathbb{Q} represents the probabilities of the various states,

and who measures wealth in terms of the asset $A^{(i)}$ (the *risk-free* asset). For this reason, \mathbb{Q} is called the *risk-neutral* probability measure for $A^{(i)}$ (risk-adjusted might be a better term, since it's the hypothetical investor who is risk-neutral, not the probabilities).

Close inspection will show that the risk-neutral probabilities do not depend on the choice of \mathbb{P} ; they depend only on prices, and are quite unrelated to the *actual* probabilities. In fact,

$$\mathbb{Q}(\{\omega_j\}) = Z_j/D(\omega_j) = Z_j A_1^{(i)}(\omega_j)/A_0^{(i)}.$$

1.3 Example: Delta Airlines

At one point in early 2005, Delta's 10% notes due 2008 traded at a price of \$71.50. At around the same time, a zero-coupon Treasury STRIP also due in 2008 had a price of 90:20, or $\$90 + 20/32 = \90.625 . We can use these prices to infer the risk-neutral probability of bankruptcy for Delta.

We set up a single-period, two-asset economy with $A^{(1)}$ = Delta and $A^{(2)}$ = the Treasury STRIP. The time $t = 0$ corresponds to early 2005, and

$$A_0 = \begin{bmatrix} 71.50 \\ 90.625 \end{bmatrix}.$$

The time $t = 1$ corresponds to the maturity date in 2008. We set $\Omega = \{\omega_1 = \text{"Delta not bankrupt"}, \omega_2 = \text{"Delta bankrupt"}\}$; the STRIP matures with a face value of \$100 in each state. The Delta note is more complicated.

Strictly, we need time steps for each of the payment dates of the interest on the Delta note; to keep things simple, we'll assume that the interest is all paid at maturity, so the value of the note is \$130 in state ω_1 , and some lesser amount Δ in ω_2 . So

$$A_1 = \begin{bmatrix} 130 & \Delta \\ 100 & 100 \end{bmatrix}$$

To verify that the economy is arbitrage-free, we calculate the pricing kernel Z , which satisfies $A_0 = A_1 Z$:

$$Z = \begin{bmatrix} 130 & \Delta \\ 100 & 100 \end{bmatrix}^{-1} \begin{bmatrix} 71.50 \\ 90.625 \end{bmatrix} = \frac{1}{13000 - 100\Delta} \begin{bmatrix} 7150 - 90.625\Delta \\ 4631.25 \end{bmatrix}.$$

The value of the Delta note in default is certainly less than \$130, so the scalar multiplier is positive, so for the economy to be arbitrage-free, we must

have $\Delta < 7150/90.625 = 78.90$. Otherwise, we could buy \$100 in face value of the Delta note and sell short Δ in face value of the STRIP, realizing $\$(0.90625\Delta - 71.50)$ up front, and in either state the return on the Delta note would be enough to repay the principal on the STRIP.

To be conservative, we would put $\Delta = 0$, whence

$$Z = \begin{bmatrix} 0.55 \\ 0.35625 \end{bmatrix}.$$

We take the STRIP as the reference asset, and since its value is the same in both states, the discount factor D is the constant 0.90625 and $\mathbb{Q}(\{\omega_1\}) = .6069$, $\mathbb{Q}(\{\omega_2\}) = .3931$. That is, the risk-neutral default probability is 39.21%. This compares quite well with the historical default frequency of CC-rated entities of 41.79%.

A less conservative calculation would recognize that $\Delta > 0$; a study by Moody's showed that senior unsecured debt such as this has a value in default that varies widely, but averages quite close to 50% of the nominal value, or $\Delta = 65$. With this value,

$$Z = \begin{bmatrix} 0.19375 \\ 0.7125 \end{bmatrix}.$$

and $\mathbb{Q}(\{\omega_2\}) = .7862$. So with this more realistic assumption, the risk-neutral probability of 78.62% is closer to twice the "actual" probability.

1.3.1 Epilogue

On September 14, 2005, Delta, the nation's No. 3 airline, filed for bankruptcy. Northwest, the No. 4 carrier, followed within minutes.