

1 Short Term Variability

Short term market variability—on a time scale from one day to one or two weeks—imposes risk on market participants. One aspect of this risk is that adverse changes in market conditions may result in unacceptable losses, that may lead to the dissolution of a “desk” or to the bankruptcy of an institution. These are the direct impacts of *market risk* on the institution. A second aspect, less direct but no less dangerous, is that such market risk may lead to the default of one of the institution’s *counterparties* and consequent loss to the institution. This is called *credit risk*. We deal first with direct market risk.

1.1 Market Risk

Suppose that a bank has a portfolio of financial instruments subject to market risk. These may be actual securities or derivatives. Write $V_i(t, m)$ for the value of the i th instrument at time t when the market conditions are as summarized in m .

Notes:

- m is multivariate: it must contain yield curves and exchange rates for all currencies represented in the portfolio, volatilities of any underlying variable involved in option-like instruments, and equity (stock-market) variables if necessary.
- Market conditions at time t will be denoted m_t ; thus the value of the i th instrument at time t is actually $V_i(t, m_t)$.
- $V_i(t, m)$ depends on time t directly as well as through m , because for instance the remaining maturity decreases as t increases.
- $V_i(t, m)$ is a nonlinear function of m —mildly so for simple instruments like bonds and swaps, considerably more so for options and option-like instruments, especially when the “exercise date” of the option is near.

1.2 P&L distribution

If (and this is a major assumption) no changes are made to the portfolio between times t (representing the “present”) and $t + \delta t$, then the change in

value of the portfolio is

$$\delta V = \sum_i V_i(t + \delta t, m_{t+\delta t}) - V_i(t, m_t).$$

The change δV represents net profit and loss, or P&L. Questions about the “level of risk” in the portfolio may be answered by using the probability distribution of δV for an appropriate δt , conditional on $m_u, u \leq t$. Typically δt is one day, and the only aspect of this distribution that is used is the “Value at Risk”, namely the (magnitude of the) lower $100\alpha\%$ percentile:

$$P(\delta V > -\text{VaR}) = 1 - \alpha.$$

That is, the VaR is the level of loss over δt days that will be exceeded with a probability of α . Typically, δt is one or ten days, and α is .05 or .01. On the assumption, again, that the positions are not changed during the day, we would expect that on one out of 20 days, the losses would actually exceed the one-day 5% VaR.

1.3 Computing The Distribution of P&L

How can we calculate this distribution? An exact analytic approach is infeasible. The simplest approach is:

- approximate $V_i(t, m)$ by a first order expansion in δm and δt ;
- approximate the joint (conditional) distribution of δm by a multivariate Gaussian;
- estimate the necessary variances and covariances (means over short times may be assumed to be zero); and
- compute the resulting Gaussian distribution of δV , which of course has mean zero and is therefore characterized by its standard deviation.

The value functions $V_i(t, m)$ are rarely given explicitly, whence the required partial derivatives must be obtained numerically, by evaluating V under perturbed market conditions close to m_t . The time derivative $\partial V/\partial t$ is typically small enough to be ignored.

This approach is of course subject to the criticism that either or both of the approximations may be too inexact. If either is dropped, the resulting distribution of δV becomes nonGaussian. The only approaches that are currently feasible involve simulation:

- draw a sample m from the (conditional) distribution of $m_{t+\delta t}$;
- value every instrument at the sampled m ; and
- save the resulting value of the portfolio.

After enough iterations, the saved values may be used to estimate any functional of the distribution of δV .

Several choices must be made:

- The conditional distribution of δm may be assumed to belong to some parametric family. Alternatively, parametric assumptions may be avoided by bootstrap methods: resampling δm from an empirical distribution of observed changes in past data.
- The valuations may be “exact”, that is, obtained from the same model used to price the instruments, or may be approximations.
- The portfolio valuations may be used directly to obtain empirical percentiles, or a parametric family may be fitted to them; for instance, the left tail of the distribution could be fitted by a Generalized Pareto density:

$$f(x; \xi, \beta) = \frac{1}{\beta} \left(1 - \xi \frac{x}{\beta}\right)^{\xi^{-1}-1}.$$

1.4 Measures of Risk

As noted above, for risk management purposes, the P&L distribution is usually summarized by a single number that quantifies the magnitude of the risk, and this *risk measure* is typically VaR, a quantile.

Artzner, Delbaen, Eber, and Heath (1998; link on the course home page) introduced four criteria that a risk measure should have, and call a measure that satisfies all four *coherent*. The concept is also discussed by Duffie and Singleton in Section 2.4.5. The setting is a sample space Ω of outcomes, and portfolio payoffs (random variables) $X(\omega), Y(\omega), \dots$, and a risk measure $m(X)$ that associates a real-valued risk quantity with any portfolio payoff. The criteria are:

- Subadditivity: For any portfolio payoffs X and Y ,

$$m(X + Y) \leq m(X) + m(Y);$$

- Homogeneity: For any number $\theta > 0$, $m(\theta X) = \theta m(X)$;
- Monotonicity: $m(X) \geq m(Y)$ if $X \leq Y$;
- Risk-free condition: $m(X + k) = m(X) - k$, for any constant k .

Of these, Homogeneity is debatable: if θ is large enough, the risk in the position may become unacceptable; however, it is innocuous for moderate values. Subadditivity is necessary to recognize diversification: combining two portfolios cannot increase the combined risk.

Subadditivity and Homogeneity imply:

- Convexity: For any $0 \leq \theta \leq 1$,

$$m(\theta X + (1 - \theta)Y) \leq \theta m(X) + (1 - \theta)m(Y).$$

In some recent work, Subadditivity and Homogeneity are replaced by the weaker assumption of Convexity.

Note that no probability measure on Ω has been introduced: the criteria for coherence do not require one. However, when we have one, using it to propose a candidate $m(\cdot)$ seems reasonable. The formal definition of VaR_α is

$$\text{VaR}_\alpha(X) = \sup\{x : P(-X \geq x) > \alpha\}.$$

If X is continuous with a strictly positive density function, $\text{VaR}_\alpha(X)$ is the upper α -quantile of $-X$; that is, the unique solution of the equation

$$P(-X \geq v) = \alpha.$$

The formal definition continues to work when this equation has either no solution or many solutions.

This clearly satisfies Homogeneity, Monotonicity, and the Risk-free condition, but fails Subadditivity. To see this, suppose that X and Y are independent and each is $-a < 0$ with probability p and 0 with probability $1 - p$, and $\alpha/2 < p < \alpha$. Then $\text{VaR}_\alpha(X) = \text{VaR}_\alpha(Y) = 0$, but $\text{VaR}_\alpha(X + Y) = a$. More realistically, suppose that X and Y are payoffs on short positions in deep-out-of-the-money options that expire before the end of the period of interest. Then again, both X and Y are zero with high probability, but have possibly long left tails on the negative half-line. If the probability of a zero payout is close enough to 1, VaR is zero for each option separately, but may be positive for their sum.

One simple way to construct a coherent risk measure is to use a family \mathcal{Q} of probability measures, and quantify risk by the worst-case expected loss:

$$m_{\mathcal{Q}}(X) = \sup_{Q \in \mathcal{Q}} E_Q(-X).$$

The general idea is that the distributions in \mathcal{Q} should represent various forms of stress, without deviating too far from what is realistic. Any such $m_{\mathcal{Q}}(\cdot)$ is easily shown to be coherent. The converse is also true, but harder to prove: for any coherent risk measure $m(\cdot)$ there exists such a family \mathcal{Q} .

If we have a model that provides a distribution P , we could take the family \mathcal{Q} to be those measures that are boundedly stressed versions of P :

$$\mathcal{Q} = \left\{ Q : \frac{dQ}{dP} \leq k \right\}$$

for some constant $k > 1$, which we write as $1/\alpha$ for some α , $0 < \alpha < 1$. If A is any event, this means that $Q(A) \leq k \times P(A)$; that is, $Q(A)$ might be larger than $P(A)$, but only within limits.

We can write any such Q as $dQ(\omega) = \Lambda(\omega)dP(\omega)$, where $0 \leq \Lambda(\omega) \leq k$ and

$$1 = \int dQ(\omega) = \int \Lambda(\omega)dP(\omega),$$

or in random variable notation, $E_P(\Lambda) = 1$. Then we can write

$$m(X) = \sup_{\Lambda: 0 \leq \Lambda \leq k, E(\Lambda)=1} E_P(-\Lambda X).$$

Clearly, to maximize $E_P(\Lambda X)$ subject to these conditions, we should make Λ as large as possible (k) where X is large and negative, and zero otherwise. To keep $E_P(\Lambda) = 1$, the set where $\Lambda = k$ must have probability (at most) $1/k = \alpha$. If X has a continuous distribution with a positive density, this set is just $\{\omega : -X(\omega) \geq \text{VaR}_{\alpha}(X)\}$, and

$$m(X) = \frac{1}{\alpha} E_P[-X \mathbb{1}_{\{-X \geq \text{VaR}_{\alpha}(X)\}}] = E_P(-X | -X \geq \text{VaR}_{\alpha}(X)). \quad (1)$$

The right hand side of (1) is known as the Expected Shortfall, or $\text{ES}_{\alpha}(X)$.

In general, the worst-case Λ may not be unique, but one version satisfies

$$\Lambda = \begin{cases} k & \text{if } -X > \text{VaR}_{\alpha}(X), \\ 0 & \text{if } -X < \text{VaR}_{\alpha}(X), \\ k' & \text{if } -X = \text{VaR}_{\alpha}(X), \end{cases}$$

where $0 \leq k' \leq k$ is determined by the requirement that

$$1 = E_P(\Lambda) = k \times P[-X > \text{VaR}_\alpha(X)] + k' \times P[-X = \text{VaR}_\alpha(X)]$$

whence

$$k' = \frac{1 - k \times P[-X > \text{VaR}_\alpha(X)]}{P[-X = \text{VaR}_\alpha(X)]} = \frac{\alpha - P[-X > \text{VaR}_\alpha(X)]}{\alpha \times P[-X = \text{VaR}_\alpha(X)]}$$

and

$$m(X) = \frac{1}{\alpha} \{E_P[-X \mathbb{1}_{\{-X > \text{VaR}_\alpha(X)\}}] + \{\alpha - P[-X > \text{VaR}_\alpha(X)]\} \text{VaR}_\alpha(X)\}$$

The resulting risk measure $m(X)$ is the same as the Expected Shortfall only if $P[-X \geq \text{VaR}_\alpha(X)] = \alpha$; nevertheless, it is often referred to as the Expected Shortfall. It may be shown that

$$m(X) = \frac{1}{\alpha} \int_0^\alpha \text{VaR}_{\alpha'}(X) d\alpha',$$

whence it is also known as the Average VaR or AVaR $_\alpha$.

Note that

$$\begin{aligned} \alpha [m(X) - \text{VaR}_\alpha(X)] &= E_P\{[-X - \text{VaR}_\alpha(X)] \mathbb{1}_{\{-X > \text{VaR}_\alpha(X)\}}\} \\ &= E_P\{\max[-X - \text{VaR}_\alpha(X), 0]\} \\ &= E_P[\text{amount of loss in excess of } \text{VaR}_\alpha(X)]. \end{aligned}$$

That is, $m(X)$ exceeds $\text{VaR}_\alpha(X)$ by an amount that is proportional to the expected value of losses in excess of $\text{VaR}_\alpha(X)$. This is the actuarially fair premium for insurance against those excess losses.

1.5 Market Risk Capital Requirements

VaR as described earlier is usually calculated for an institution's own internal risk management purposes. In recent years, supervisors of financial institutions have begun to recognize that the institutions' capabilities for measuring the degree of risk that they undertake is better than those of the supervisors themselves. There has been a corresponding move (by the Bank for International Settlements in Basel as the international coordinator and by some national supervisors in the European Community and the U.S.)

to allow each institution to play some role in the calculations of regulatory capital for covering market risk. The move generally refers to the “internal models” developed by the institutions.

Since capital must not be completely consumed by losses except under extraordinary circumstances, the criteria for defining regulatory capital are more stringent than for conventional VaR. The principal differences are:

- the capital requirement is based on potential market losses over a 10-day *holding period*, instead of a single day;
- the pertinent quantile is the 99th percentile of the distribution of losses, instead of the 95th;
- the required capital is the greater of
 - a multiple (between 3 and 4; see Section 1.5.1) of the average value of this percentile over the previous 60 days’ calculations; and
 - the current calculation, for the previous day’s portfolio.

The regulators have also restricted the institutions’ freedom as to how distributions are estimated. For instance, volatility *must* be estimated as the unweighted variance of one year’s worth of historical data; more sophisticated methods such as GARCH are prohibited. Also, certain correlations are required to be set equal to 1, for conservatism.

1.5.1 Backtesting

Testing the validity of VaR calculations is of course a normal part of implementing and maintaining such a capability. When a VaR model is to be used as a basis for determining regulatory capital, such testing has to be formalized and codified. One current proposal is based on *backtesting* the calculation over the recent past.

Backtesting proceeds as follows. Take a given portfolio of instruments, say the “current” book. For each of the most recent 250 trading days:

- value the book;
- re-estimate all parameters in the VaR model based on prior data;
- calculate the VaR of the book based on these re-estimated parameters;

- revalue the book one day (or one holding period) later, and subtract to obtain the actual change in value;
- determine whether the change exceeded the VaR (in the direction of loss).

The number of occasions on which the VaR was exceeded should be around $250 \times .01 = 2.5$ occasions. The outcomes are classified as follows.

Green zone: Finding up to 4 exceedances is viewed as bad luck.

Yellow zone: More than 4 but no more than 9 is viewed as requiring an empirical adjustment, which consists of scaling the multiplier of 3 progressively up to 4.

Red zone: Finding 10 or more exceedances is bad news.

1.6 Credit Risk and the Use of Collateral

Short term market variability may cause credit risk as well as market risk. As laid out above, *market risk* is the threat to an institution of losses caused by the impact of adverse market changes on its *own* portfolio. *Credit risk* is the indirect threat that adverse market moves may cause losses in *another* institution's portfolio, driving that institution into insolvency. If the first institution is *exposed to* the insolvent party, that is if the insolvency triggers an obligation of the insolvent party to make a payment, then the insolvency may cause economic loss for the first institution.

Many parties now use the pledging of *collateral* to manage this short term credit risk. The party with the obligation to pay delivers securities (Treasury issues, for instance) to the other party. The securities remain the property of the pledgor, who also continues to receive coupon payments, exercise voting rights, and so on. The pledgee holds a *first security interest* in the securities, however, which gives it the right to assume ownership in specified circumstances. *In favorable legal jurisdictions*, this right supersedes the right of a bankruptcy court to control the disposition of the assets of an insolvent entity.

The question arises of how much collateral, and of what kind, constitutes an adequate protection. This question may be answered using concepts from the calculation of Value at Risk. Specifically, suppose that the concerned institution (“us”) has a *portfolio* of n transactions with another

entity (“them”), with values $V_i(t, m)$ at time t under market conditions m . We assume that

$$V(t, m_t) = \sum_{i=1}^n V_i(t, m_t) > 0$$

(the portfolio is, on a net basis, *in the money* to us at time t). If “we” hold as collateral n' securities with values $V'_i(t, m)$, our unsecured exposure to them is

$$V(t, m_t) - \sum_{i=1}^{n'} V'_i(t, m_t) = \sum_{i=1}^{n+n'} V_i(t, m_t) = V_U(t, m_t),$$

where we write $V_{n+i}(\cdot) = -V'_i(\cdot)$, $i = 1, 2, \dots, n'$. This is the value of a hypothetical portfolio consisting of the existing deals and *short* positions in the collateral securities. The collateral is satisfactory if the hypothetical portfolio has negative value now and a high probability of staying negative for an acceptable period of time. That is, we want

$$V_U(t, m_t) \leq 0$$

and

$$P(V_U(t + \delta t, m_{t+\delta t}) = V_U(t, m_t) + \delta V \leq 0) \geq 1 - \epsilon.$$

Strictly speaking, we should require

$$P(V_U(t + u, m_{t+u}) \leq 0, 0 \leq u \leq \delta t) \geq 1 - \epsilon,$$

but the additional complication is not worthwhile.

Note that

$$P(V_U(t + \delta t, m_{t+\delta t}) = V_U(t, m_t) + \delta V \leq 0) = P(\delta V \leq -V_U(t, m_t))$$

is essentially the same quantity that we needed to find the VaR. The differences are:

- δt may be longer than one day. One week is common, and one month also occurs. Longer intervals increase the variability of δV and thus increase the *over-collateralization*, but mean less frequent transfers of collateral.
- ϵ may not be .05. “We” may need a degree of protection ranging from
 - extreme ($\epsilon = .001$ or less), if our credit rating is critically dependent on avoiding loss, to
 - relaxed ($\epsilon = .15$, “ 1σ ”) if we are not in a position to ask for more.