E5. We have an iid sample $Y_1, \ldots, Y_n$ from a $N(\mu, \sigma^2)$ distribution and a complicated estimator $\hat{g}$ of $g(\mu, \sigma)$. Give a method for checking to see if $\hat{g}$ is a function solely of the complete sufficient statistic $T = (\overline{Y}, S)$. If $\hat{g}$ is not a function only of $T$, suggest a computational-based method for improving it.

E6. John Monahan had a consulting problem where on night 1, $X_1$ out of $N$ insects were eaten. On the second night, $X_2$ out of $N - X_1$ insects were eaten. Assuming that the probability that an insect is eaten stays constant at $p$, the goal is to estimate $p$. We can compute the likelihood using the fact that $X_1$ and $X_2|X_1$ are binomial or that jointly $(X_1, X_2, N - X_1 - X_2)$ is multinomial($N, p_1 = p, p_2 = p(1 - p), p_3 = 1 - p_1 - p_2$).

a) Find the maximum likelihood estimator of $p$.

b) Find an unbiased estimator of $p$. What happens when we try to condition on $(X_1, X_2)$?

c) Show that this particular multinomial family is a curved exponential family and find an unbiased estimator $T$ of 0 (thus the minimal sufficient statistic is not complete). (Hint: use the binomial example from class about the UMVU estimator of $p(1 - p)$ and note that marginally $X_2$ is binomial($N, p(1 - p)$).)

d) Using b) and c), give a general form for unbiased estimators of $p$. (Hint: just form a linear combination that is unbiased.)

E7. (Problem 4.25, p. 144 of TSH). Let $X, Y$ be independently distributed according to the negative binomial distributions $Nb(p_1, m)$ and $Nb(p_2, n)$, respectively. For example, $X$ has probability mass function

$$P(X = x) = \binom{m + x - 1}{x} p_1^m (1 - p_1)^x, \quad x = 0, 1, 2, \ldots$$

a) Let $q_i = 1 - p_i$. Show that there exists a UMP unbiased test for $H : \theta = q_2/q_1 \leq \theta_0$ and hence in particular for $H' : p_1 \leq p_2$.

b) Determine the conditional distribution required for testing $H'$ when $m = n = 1$. 