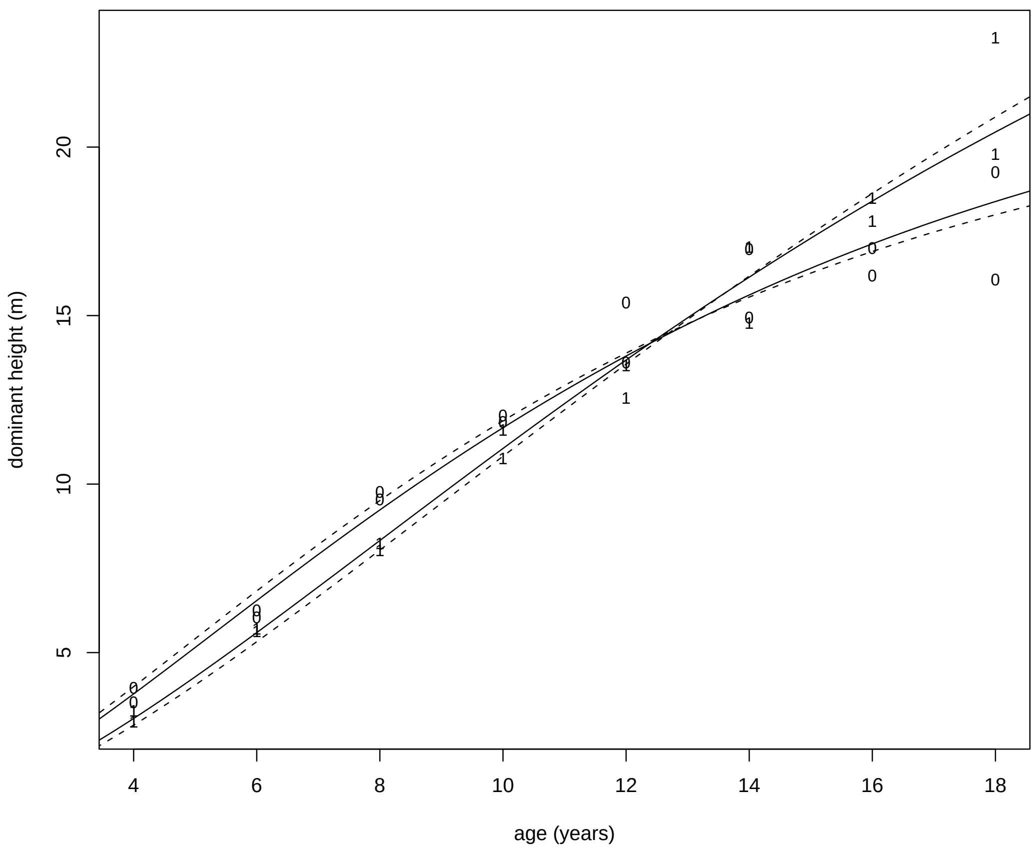
ST 762, HOMEWORK 2 SOLUTIONS, FALL 2009

1. (a) Here are plots of the data by site preparation treatment (with the fits for part (c) superimposed, too).



Because there is replication at each x value, we can get a sense of whether there is nonconstant variance visually with ease. It does seem as though the replicates at the lower times (lower values of response) are less variable than those at higher concentrations for the most part. This is consistent with variance that increases as the mean response increases.

(b) See program and output.

(c) The model seems to fit the data well. The solid lines show the OLS fit obtained when setting the treatment indicator to 0 (control) and 1 (chop-bed-burn), respectively; the dashed lines are the corresponding GLS fit. The two fits seem pretty similar. We'll see later in the course why the GLS estimator is nonetheless to be preferred on theoretical grounds.

2. (a) The model is

$$f(\boldsymbol{x}_j^T\boldsymbol{\beta}) = \exp(\boldsymbol{x}_j^T\boldsymbol{\beta}).$$

This is a loglinear model often used to model the mean of count data. Under the Poisson assumption, the likelihood is

$$\prod_{j=1}^{n} f(\boldsymbol{x}_j^T\boldsymbol{\beta})^{Y_j}\exp\{-f(\boldsymbol{x}_j^T\boldsymbol{\beta})\}/Y_j!.$$

Taking logs and differentiating with respect to $\boldsymbol{\beta}$ gives (details are not shown here; you should have done this)

$$\sum_{j=1}^{n}\{f(\boldsymbol{x}_j^T\boldsymbol{\beta})\}^{-1}\{Y_j - f(\boldsymbol{x}_j^T\boldsymbol{\beta})\}f_\beta(\boldsymbol{x}_j^T\boldsymbol{\beta}) = \mathbf{0}.$$

Of course, the reciprocal term in braces is the variance. This is of course of the GLS form.

(b) Writing $f_j = f(\boldsymbol{x}_j^T\boldsymbol{\beta})$, we have $f'(\eta) = f(\eta)$, and thus $\boldsymbol{Q}(\boldsymbol{\beta}) = \mathrm{diag}(f_j)$ and $\boldsymbol{W}(\boldsymbol{\beta}) = \mathrm{diag}(1/f_j)$, so that $\boldsymbol{W}_* = \mathrm{diag}(f_j)$. Moreover, $\boldsymbol{Z}_* = \boldsymbol{X}_*\boldsymbol{\beta} + \boldsymbol{Q}^{-1}(\boldsymbol{Y} - \boldsymbol{f})$. Now the jth row of $\boldsymbol{Z}_*$ is

$$\boldsymbol{Z}_{*j} = \boldsymbol{x}_j^T\boldsymbol{\beta} + \frac{Y_j - f_j}{f_j},$$

and if $\eta_j = \boldsymbol{x}_j^T\boldsymbol{\beta}$, then $\eta_j = \log(f_j) = \boldsymbol{x}_j^T\boldsymbol{\beta}$. Thus, $\boldsymbol{Z}_{*j}$ may be written in terms of f_j as

$$\boldsymbol{Z}_{*j} = \log(f_j) + \frac{Y_j - f_j}{f_j},$$

and we may insert in place of f_j in this expression the initial guess $f_{*j} = (Y_j + \overline{Y})/2$ and get a starting value for $\boldsymbol{\beta}$ by solving

$$\boldsymbol{\beta} = \{\boldsymbol{X}_*^T\mathrm{diag}(f_{*j})\boldsymbol{X}_*\}^{-1}\boldsymbol{X}_*^T\mathrm{diag}(f_{*j})\boldsymbol{Z}_*,$$

where $\boldsymbol{Z}_*$ has jth element

$$\log(f_{*j}) + \frac{Y_j - f_{*j}}{f_{*j}}.$$

(c) See program and output.

(d) See program and output.

(e) Let $\boldsymbol{x}_0 = (1, 0, 0, 55, 1, 1)^T$ and $\mu_0 = \exp(\beta_1 + \beta_2 x_{j1} + \beta_3 x_{j2} + \beta_4 x_{j3} + \beta_5 x_{j4} + \beta_6 x_{j5})$. Then we want

$$P(Y \le 2) = P(Y = 0) + P(Y = 1) + P(Y = 2) = \exp(-\mu_0)(1 + \mu_0 + \mu_0^2/2).$$

Substituting in the estimates gives the point estimate 0.38 (see the program and output).