# ST 762, HOMEWORK 4 EXTRA PROBLEMS, FALL 2007

1. Suppose that $Y_j$, $j = 1, \ldots, n$, are independent, and suppose we assume that

$$E(Y_j) = mh(\beta) = m\left(\frac{e^\beta}{1+e^\beta}\right), \quad \text{var}(Y_j) = mh(\beta)\{1-h(\beta)\}, \quad j = 1, \ldots, n, \tag{1}$$

where $m \geq 1$ is an integer.

Note: for each part (a)–(d), you must derive the result explicitly; *do not* just plug into theorems proved in class.

(a) Under model (1), find the GLS estimator $\hat\beta$ for $\beta$ for $C = \infty$ and, assuming that (1) is the *correct* model for mean and variance of $Y_j$, show that if $\beta_0$ is the true value for $\beta$, $\hat\beta \xrightarrow{p} \beta_0$ (i.e. $\hat\beta$ is consistent).

(b) Under the conditions of (a), derive the limiting distribution of $n^{1/2}(\hat\beta - \beta_0)$ and show that it is of the form dictated by the "folklore" theorem. (*Hint:* You may have to invoke some kind of approximation in order to use familiar theorems.)

(c) Now suppose that we compute the GLS estimator *assuming* (1), but *in truth*, the model is

$$E(Y_j) = mh(\beta), \quad \text{var}(Y_j) = mh(\beta)\{1-h(\beta)\}\{1+\theta(m-1)\} \tag{2}$$

for some known $\theta > 0$. Is $\hat\beta$ consistent for the true value $\beta_0$ in model (2)? Show explicitly why or why not.

(d) Under the conditions of (c), derive the limiting distribution of $n^{1/2}(\hat\beta - \beta_0)$ and verify that it is of the form on page 224.

2. Consider the usual mean-variance model

$$E(Y_j|\boldsymbol{x}_j) = f(\boldsymbol{x}_j, \boldsymbol{\beta}), \quad \text{var}(Y_j|\boldsymbol{x}_j) = \sigma^2 g^2(\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{x}_j), \tag{3}$$

where $\boldsymbol{\beta}$ is $(p \times 1)$, $\boldsymbol{\theta}$ is $(q \times 1)$, $j = 1, \ldots, n$, and $(Y_j, \boldsymbol{x}_j)$ are independent across $j$. Suppose that $E(Y_j|\boldsymbol{x}_j)$ and $\text{var}(Y_j|\boldsymbol{x}_j)$ are correctly specified in (3), and that $\boldsymbol{\beta}_0$, $\sigma_0 > 0$, and $\boldsymbol{\theta}_0$ represent the true values of the parameters. Define as usual

$$\epsilon_j = \frac{Y_j - f_{0j}}{\sigma_0 g_{0j}},$$

where here we use the shorthand notation in the notes indicating evaluation at the true parameter values.

Suppose we assume that $Y_j|\boldsymbol{x}_j$ has coefficient of skewness $\zeta_j$ and coefficient of excess kurtosis $\kappa_j$, where $\zeta_j$ and $\kappa_j$ are a set of constants, and decide to estimate $\boldsymbol{\beta}$, $\sigma$, and $\boldsymbol{\theta}$ jointly by $\hat{\boldsymbol{\beta}}$, $\hat\sigma$, and $\hat{\boldsymbol{\theta}}$ solving equation (10.2) on page 245 of the notes.

In the notes, we derived the limiting distribution as $n \rightarrow \infty$ of $n^{1/2}\{(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^T/\sigma_0, (\hat\sigma - \sigma_0)/\sigma_0, (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^T\}^T$ when $\zeta_j \equiv 0$ and $\kappa_j \equiv \kappa$ (same for all $j$). In this problem, you will find out that the general case is not so hard after all by carrying out similar arguments for the situation of arbitrary assumed values $\zeta_j$ and $\kappa_j$.

*In all parts of the problem below, you should simplify your answers to the best of your ability, providing results that are as concise as possible. Behave as if you were preparing these* *results for a dissertation, journal article, or other report, where clarity and conciseness are demanded. Simplification will also allow you to see the solutions easily.*

(a) Define $\boldsymbol{H}$ to be the $(n \times n)$ diagonal matrix with diagonal elements $(2+\kappa_j - \zeta_j^2)^{-1}$, define $\boldsymbol{Z}$ to be the $(n \times n)$ diagonal matrix with diagonal elements $\zeta_j$, and define $\boldsymbol{X}$, $\boldsymbol{W}$, $\boldsymbol{Q}$, and $\boldsymbol{R}$, and other quantities as in the notes. Let

$$\boldsymbol{X}_* = \boldsymbol{Z}\boldsymbol{H}^{1/2}\boldsymbol{X}, \quad \boldsymbol{Q}_* = \boldsymbol{H}^{1/2}\boldsymbol{Q}, \quad \boldsymbol{R}_* = \boldsymbol{H}^{1/2}\boldsymbol{R},$$

where $\boldsymbol{H}^{1/2}$ is the diagonal matrix whose elements are the square roots of those in $\boldsymbol{H}$.

Analogous to the what is presented in Section 10.2 of the notes, find $\boldsymbol{A}_n^*$ and $\boldsymbol{C}_n^*$ satisfying

$$\boldsymbol{A}_n^* n^{1/2} \begin{pmatrix} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)/\sigma_0 \\ (\hat{\sigma} - \sigma_0)/\sigma_0 \\ \hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0 \end{pmatrix} \approx \boldsymbol{C}_n^*.$$

Express $\boldsymbol{A}_n^*$ in terms of $\boldsymbol{X}$, $\boldsymbol{W}$, $\boldsymbol{X}_*$, $\boldsymbol{Q}_*$, and $\boldsymbol{R}_*$. Express $\boldsymbol{C}_n^*$ in terms of the $\epsilon_j$ and quantities defined in the notes.

(b) Suppose that, amazingly, $\zeta_j$ and $\kappa_j$ are *correctly specified* for all $j$. Under these conditions, find an approximation for large $n$ to the matrix $\boldsymbol{\Sigma}$ satisfying

$$n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)/\sigma_0 \xrightarrow{L} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}).$$

Express your approximation in terms of $\boldsymbol{X}$, $\boldsymbol{W}$, $\boldsymbol{R}_*$, and $\boldsymbol{P}_* = \boldsymbol{I} - \boldsymbol{Q}_*(\boldsymbol{Q}_*^T\boldsymbol{Q}_*)^{-1}\boldsymbol{Q}_*^T$.

(c) Under the conditions of (b), show explicitly that $\hat{\boldsymbol{\beta}}$ found by solving (10.2) is at least as efficient as the GLS estimator $\hat{\boldsymbol{\beta}}_{GLS}$ found assuming (3) is correct.

(d) Show explicitly that your result in (b) reduces to the analogous result in the notes (identify this result explicitly by page number) when $\zeta_j$ and $\kappa_j$ are *correctly chosen* to be $\zeta_j \equiv 0$ and $\kappa_j \equiv \kappa$, a constant for all $j$.

(e) Suppose that the true variance model is $g(\boldsymbol{\beta}, \boldsymbol{\theta}, \boldsymbol{x}_j) = f^{\theta}(\boldsymbol{x}_j, \boldsymbol{\beta})$, with true value $\theta_0 \geq 0$, and we specify this model correctly in setting up the equations, as above. Suppose further that the constants $\zeta_j$ and $\kappa_j$ we choose are such that $\zeta_j \equiv \zeta = a\sigma_0$ and $\kappa_j \equiv \kappa = b\sigma_0^2$ for some constants $a$ and $b$, and suppose these values are *correct*. Find an approximation for large $n$ to the matrix $\boldsymbol{\Sigma}$ satisfying $n^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)/\sigma_0 \xrightarrow{L} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$, and express your approximation in terms of $\boldsymbol{X}$, $\boldsymbol{W}$, $\boldsymbol{Z}$, $\boldsymbol{P}$ (where $\boldsymbol{P}$ is as defined in the notes on page 253), and any other matrices you need to define.

(f) In (e), find values of $a$, $b$, and $\theta_0$ such that the approximation to $\boldsymbol{\Sigma}$ is exactly equal to the approximation to the large sample covariance matrix of $\hat{\boldsymbol{\beta}}_{GLS}$. *Explain what you think is going on.*

3. Suppose that we *know* that $E(Y_j) = f_j(\beta)$ (so you may assume this throughout), where

$$\begin{aligned} f_j(\beta) &= \beta, \quad j = 1, \ldots, m \\ &= 1/\beta, \quad j = m+1, \ldots, n \end{aligned}$$

where $n = 2m$ and $\beta > 1$. Define as usual $\beta_0$ to be the true value of $\beta$, and let

$$\epsilon_j = \frac{y_j - f_j(\beta_0)}{\{\text{var}(Y_j)\}^{1/2}},$$

where $\text{var}(Y_j)$ is the *true* variance of $Y_j$. *In solving the problems below, you must derive the results yourself – do not try to "plug in" to results proved in general in the notes.*

(a) Suppose we believe that

$$\text{var}(Y_j) = \sigma^2 f_j(\beta), \quad E(\epsilon_j^3) = 0, \quad \text{var}(\epsilon_j^2) = 2 + \kappa,$$

where $\sigma > 0$, and $\kappa$ is a *known* constant and is the same for all $j$. Write down the "optimal" quadratic estimating equation for $\beta$ under these assumptions.

(b) Define $T_1 = m^{-1}\sum_1^m Y_j$, $T_2 = m^{-1}\sum_{m+1}^n Y_j$, $Z_1 = m^{-1}\sum_1^m Y_j^2$, and $Z_2 = m^{-1}\sum_{m+1}^n Y_j^2$. Find the estimator for $\beta$, $\widehat{\beta}_Q$, solving the equations in (a) in terms of $T_1$, $T_2$, $Z_1$, and $Z_2$.

(c) Assuming that the assumptions in (a) are *exactly correct*, and the true value of $\sigma$ is $\sigma_0$, show that $\widehat{\beta}_Q$ is consistent for $\beta_0$ as $m \rightarrow \infty$.

(d) Suppose now that we believe

$$\text{var}(Y_j) = \sigma^2 f_j(\beta), \quad E(\epsilon_j^3) = 0, \quad \text{var}(\epsilon_j^2) = 2, \tag{4}$$

for some $\sigma > 0$; i.e., the same as in (a) but with $\kappa = 0$. Write down the form of $\widehat{\beta}_Q$ under these conditions (you may use your results from (b) or find it directly). Call this estimator $\widehat{\beta}_Q^*$. Assuming that (4) *really is true*, and there are true values $\beta_0$ and $\sigma_0$, find the limiting distribution of $m^{1/2}(\widehat{\beta}_Q^* - \beta_0)$ as $m \rightarrow \infty$.

(e) Consider the estimator $\widehat{\beta}_Q^*$ computed under the conditions in (d). Suppose that, *in truth*, $E(\epsilon_j^3) = 0$ and $\text{var}(\epsilon_j^2) = 2$, but the variance has the form

$$\begin{aligned} \text{var}(Y_j) &= \sigma^2 \beta^\theta, \quad j = 1, \ldots, m \\ &= \sigma^2 (1/\beta)^\theta, \quad j = m+1, \ldots, n, \end{aligned}$$

with true values $\sigma_0$ and $\theta_0 > 0$. Find the limit in probability of $\widehat{\beta}_Q^*$ as $m \rightarrow \infty$. Is $\widehat{\beta}_Q^*$ consistent for $\beta_0$ in general?

(f) Assuming that $\text{var}(Y_j) = \sigma^2 f_j(\beta)$ for all $j$, find the GLS estimator of $\beta$, $\widehat{\beta}_L$, using $C = \infty$ iterations of the GLS algorithm. Express the estimator in terms of $T_1$, $T_2$, $Z_1$, and $Z_2$ defined in (b).

(g) Show that $\widehat{\beta}_L$ found in (f) is consistent for $\beta_0$ as $m \rightarrow \infty$ *regardless* of whether the variance model in (f) is the *true* model.