Moment-generating functions

This method is based on a uniqueness theorem, which states that, if two r.v. have identical moment-generating functions, the two r.v.'s possess the same probability distributions.

To use this method, we must find the moment-generating function for U and compare it with the moment-generating functions for the common discrete and continuous r.v.'s study in chapters 3 and 4. If it is identical to one of these moment-generating functions, the probability distribution of U can be identified because of the uniqueness theorem.

Uniqueness Theorem

Let $m_X(t)$ and $m_Y(t)$ denote the moment-generating functions of r.v.'s X and Y, respectively. If both moment-generating functions exist and $m_X(t) = m_Y(t)$ for all values of t, then X and Y have the same probability distribution.

The m.g.f. method is often very useful for finding the distributions fo sums of independent r.v.'s.

Theorem

Let $Y_1, Y_2, \ldots, Y_n$ be independent r.v.'s with m.g.f. $m_{Y_1}(t), m_{Y_2}(t), \cdots, m_{Y_n}(t)$ respectively. If $U = Y_1 + Y_2 + \cdots + Y_n$ then

$$m_U(t) = m_{Y_1}(t) \times m_{Y_2}(t) \times \cdots \times m_{Y_n}(t)$$

The m.g.f method can be used to establish some interesting and useful results about the distributions of some functions of normally distributed r.v.'s

Theorem Let $Y_1, Y_2, \ldots, Y_n$ be independent **normally** distributed r.v.'s with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$ and let $a_1, a_2, \ldots, a_n$ be constants. If

$$U = \sum_{i=1}^{n} a_i Y_i$$

then U is a normally distributed r.v. with

$$E(U) = \sum_{i=1}^{n} a_i \mu_i$$

and

$$V(U) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

Theorem

Let $Y_1, Y_2, \ldots, Y_n$ be independent **normally** distributed r.v.'s with $E(Y_i) = \mu_i$ and $V(Y_i) = \sigma_i^2$, for $i = 1, 2, \ldots, n$ Define Z_i by

$$Z_i = \frac{Y_i - \mu_i}{\sigma_i}$$

for $i = 1, 2, \ldots, n$.

Then,

$$\sum_{i=1}^{n} Z_i^2$$

has a χ^2 distribution with n d.f.

Steps of the moment-generating method

Let U be a function of the r.v.'s $Y_1, Y_2, \ldots, Y_n$.

1. Find the moment-generating function for U, $m_U(t)$
2. compare $m_U(t)$ with other well-known moment-generating functions. If $m_U(t) = m_V(t)$ for all values of t, then U and V have identical distributions (by uniqueness theorem).