ST 778 Fall 2007 Dr. Fuentes

Section 1 - Preliminary Set Theory:

1.1 Let Ω be an abstract set of elements w. In probability theory in general we consider the set of all possible basic outcomes of an experiment and they will constitute a set Ω known also as sample space. Each basic outcome is denoted by w. Let $A \subset \Omega$ be a subset of Ω. It consists of all those elements w which belong to A. Thus if the result of the experiment leads to a basic outcome w which belongs to A i.e. if $w \in A$, then we say that the event A has occurred. If the experimental result is an outcome $w \notin A$ then we say A^c i.e. complement of A has occurred. Thus $A^c = \{w : \ w \in \Omega, \ w \notin A\}$. Let B be a subset of A i.e. $B \subset A$. Then any $w \subset B$ also belongs to A thus we say that the event B implies A. Therefore if $B \Rightarrow A$ then $B \subset A$.

Let us now consider a family of sets or equivalently of events. Let T be a fixed set called index set. For example let $T = \{1, 2, \ldots, n\}$ then we can construct a family of sets consisting of $\{A_1, A_2, \ldots, A_n\}$ or we can denote this family by $(A_t)_{t \in T}$ where we know the index set T. The union of two sets A_1 and A_2 written by $A_1 \cup A_2 = \{w :$ where $w \in A$ or $w \in B$, including those w's which belong to both$\}$. In the language of probability theory it may be described as the occurrence of at least one of the events A_1 or A_2. Similarly the simultaneous occurrence of both events is denoted by the intersection of events $A_1 \cap A_2$. Note that $A_1 \cup A_2 = A_2 \cup A_1; \quad A_1 \cap A_2 = A_2 \cap A_1;$ also
$A_1 \cup A_2) \cup A_3 = A_1 \cup (A_2 \cup A_3)$. These are known as commutative and associative laws and these laws also hold good for intersection operation also.

Two sets A_1 and A_2 are said to be disjoint if $A_1 \cap A_2$ does not have any point. A set consisting of no points is called the null set and is denoted by $\emptyset$. In probability theory disjoint sets are known as mutually exclusive events. We also know $A_1 \cap A_1 = A_1 \cup A_1 = A_1$; $A_1 \cap \Omega = A_1$ and if $A_1 \subset A_2$ then $A_1 \cup A_2 = A_2$ and $A_1 \cap A_2 = A_1$. Further more it is easy to establish

$$A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

and

$$A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$$

$$(A_1^c)^c = A_1, \ \emptyset^c = \Omega \text{ and } \Omega^c = \emptyset.$$

$$A_1 \cup A_1^c = \Omega \text{ and if } A_1 \subset A_2 \text{ then } A^c \supset A_2^c$$

Important relationships such as Demorgan's law can also be shown to be true:

$$\begin{aligned} (A_1 \cup A_2)^c &= A_1^c \cap A_2^c \\ (A_1 \cap A_2)^c &= A_1^c \cup A_2^c \end{aligned}$$

All these operations may be viewed as the laws of set operations with corresponding interpretation in the probabilistic language of events. Even though we have defined unions and intersections of two sets, it can be easily extended to any number of sets. Thus if T be any arbitrary index sets then

$$\bigcup_{t \in T} A_t = \{w : \ w \in A_t \text{ for at least one } t \in T\}$$

$$\bigcap_{t \in T} A_t = \{w : \ w \in A_t \text{ for all } t \in T\}$$

Thus if $T = \{1, 2, \ldots, n\}$ then $\bigcup_{t \in T} A = A_1 \cup A_2 \cup \ldots \cup A_n$

and $\bigcap_{t \in T} A_t = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$.

If $T = \{1, 2, 3, \ldots\ldots\}$ then $\overset{\infty}{\underset{t \in T}{\cup}} A_n = \{w : w \in A_n \text{ for some } \underline{\text{finite } n}\}$

Note that even though we write $\overset{\infty}{\underset{n=1}{\cup}} A_n$, we are <u>not</u> postulating the existence of a set A_∞.

Similarly $\overset{\infty}{\underset{n=1}{\cap}} A_n = \{w : w \in A_n \text{ for all finite } n\}$.

If $T = \emptyset$ then $\bigcup_{t \in T = \emptyset} A_t = \emptyset$ and $\bigcap_{t \in T = \emptyset} A_t$ is defined to be Ω.

A family $\{A_t\}_{t \in T}$ is said to be a disjoint family if $A_{t1} \cap A_{t2} = \emptyset$ and also $A_{t1} \cap A_{t2}$ is denoted by $A_{t1}A_{t2}$.

In the case of disjoint family $\bigcup_{t \in T} A_t$ will be denoted by $\sum_{t \in T} A_t$

Demorgan's law holds good for any index set T, i.e.

$\left(\bigcup_{t \in T} A_t\right)^c = \bigcap_{t \in T} A_t^c$ and $\left(\bigcap_{t \in T} A_t\right)^c = \bigcup_{t \in T} A_t^c$ whether T is finite, countable infinity or uncountable infinity.

The difference between two sets A and B denoted by

$A - B = \{w : w \in A \text{ and } w \notin B\} = A - (A \cap B) = AB^c = (A \cup B) - B$.

The symmetric difference two sets A and B is denoted by

$$A \triangle B = (A - B) \cup (B - A) = (AB^c) \cup (BA^c).$$

Note $A \triangle A = \emptyset$, $A \triangle B = B \triangle A$ and also $A \cup (B \triangle C) = (A \cup B) \triangle (A \cup C)$

1.2 Limits and convergence for a sequence of sets:

<u>A detour</u>: Let us first consider limits of a sequence of real numbers. Let $\{u_n\}$ be a sequence of finite real numbers. The sequence is said to be monotone increasing if $u_{n+1} \geq u_n$. Sometimes we distinguish between monotone nondecreasing and monotone increasing sequences. If $u_{n+1} \geq u_n$ then $\{u_n\}$ is monotone nondecreasing; denote it by u_n and if $u_{n+1} > u_n$ then $\{u_n\}$ is monotone increasing; denote it by $u_n \uparrow$. Monotone increasing (nondecreasing) sequences always have a limit though it may be $+\infty$. In most situations we denote a monotone increasing or monotone nondecreasing sequence $\{u_n\}$ by the same notation $u_n \uparrow$.

A number is said to be least upper bound for a monotone nondecreasing sequence if for every $\epsilon > 0$ there exists at least one element of the sequence, say u_{n_0}, n_0 finite such that $u_{n_0} \geq a - \epsilon$. Note that this statement implies that there is infinitely many u_n's greater $a - \epsilon$ since there has to be an element $\geq a - \frac{\epsilon}{2^n}$ for every n. A monotone nondecreasing sequence with a finite lub (least upper bound) has a limit and is equal to the least upper bound. Similarly we can define a monotone decreasing (nonincreasing sequence) of real numbers and such asequence with a finite greatest lower bound has a limit and is equal to the greatest lower bound.

Now if we have an arbitrary sequence of real numbers $\{u_1\}$, then from this arbitrary sequence, we first construct two sequences based on the sequence $\{u_n\}$ according to the following scheme:

Let

$$
\begin{aligned}
u_1 &= \sup\{u_1, u_2, \ldots, u_n\} && ; && w_1 = \operatorname{Inf}\{u_1, u_2, \ldots, u_n, \ldots\} \\
u_2 &= \sup\{u_2, u_3, \ldots, \ldots\} && ; && w_2 = \operatorname{Inf}\{u_2, \ldots, u_n, \ldots\} \\
u_3 &= \sup\{u_3, u_4, \ldots, \ldots\} && ; && \\
u_n &= \sup\{u_n, u_{n+1}, \ldots\} && ; && w_n = \operatorname{Inf}\{u_n, u_{n+1}, \ldots, \ldots\}
\end{aligned}
$$

Note that u_n is monotone nonincreasing or decreasing whereas w_n is monotone nondecreasing or increasing. We shall indicate that by the notation:
$u_n \downarrow$ and $w_n \uparrow$. Now if u_n has a finite greatest lower bound (glb) $= u_0$ then $u_n \downarrow u_0$. Similarly $w_n \uparrow w_0$ where w_0 is the lub of w_n sequence.
Now $u_0 = \lim u_n = \limsup u_n$ and $w_0 = \lim w_n = \liminf u_n$.
If $u_0 = w_0$ then the common value is called the limit of the arbitrary sequence $\{u_0\}$. In fact an arbitrary sequence of real numbers can have many limit points. All the limit points will lie between $\limsup u_n$ and $\liminf u_n$. So if these two limits are equal then all the possible limits points will have the same value and in that case the common value is called the $\lim u_n$. Note that $\limsup \mathrm{u}_n$ and $\liminf u_n$ always exist though they may not be equal. When they are not equal then the limit of the sequence does not exist.

Limit of a sequence of sets:

Consider a sequence of subsets $\{A_n\}$. This sequence is said to be monotone nondecreasing if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. The limit of this sequence of sets is given by $\bigcup_{n=1}^{\infty} A_n$. Note that the limiting set $\bigcup_{n=1}^{\infty} A_n$ is a subset of Ω. A point $w \epsilon \bigcup_n A_n$ if $w \epsilon$ some A_m where m is a finite integer. But since $A_n \uparrow$ then $w \epsilon A_k$ for all $k = m, m+1, \ldots\ldots$. That is each w in the limiting set belongs to all but possibly a finite number of A_n and hence w belongs to infinitely many sets of the sequence $\{A_n\}$. Similarly the sequence $\{A_n\}$ is said to be monotone nonincreasing if $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \ldots\ldots$ (denoted by $A_n \downarrow$). The limit of the sequence of sets is given by $\bigcap_{n=1}^{\infty} A_n$. In this case a w belonging to only a finite number of sets in the sequence $\{A_n\}$ cannot belong to the limiting set. So for monotone sequences the limiting set is well defined. Consider now an arbitrary sequence of sets $\{A_n\}$. Again we shall construct from this arbitrary sequence two other sequences of subsets of Ω which will be monotone, one monotone nondecreasing and the other monotone nonincreasing.

Let
$$\begin{array}{ll}
B_1 = \bigcup_{n=1}^{\infty} A_n; & C_1 = \bigcap_{n=1}^{\infty} A_n \\
B_2 = \bigcup_{n=2}^{\infty} A_n & C_2 = \bigcap_{n=2}^{\infty} A_n \\
\vdots & \vdots \\
B_k = \bigcup_{n=k}^{\infty} A_n & C_k = \bigcap_{n=k}^{\infty} A_n \\
\vdots & \vdots
\end{array}$$

Note that $\{B_n\}$ is monotone nonincreasing as $B_1 \supseteq B_2 \supseteq B_3 \supseteq \ldots$ and similarly $\{C_n\}$ is monotone nondecreasing.

$\therefore$ The limit of $\{B_n\} = \bigcap_{n=1}^{\infty} \mathrm{B}_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$ and this is known as

lim sup A_n. Similarly the limit of $\{C_n\} = \bigcup_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} \overset{\infty}{\underset{m=n}{\underline{A}}} A_m$ and is

known as lim inf A_n. Now if these two limiting sets are equal then this common set is known as lim A_n. Note that if $A_n \uparrow$ then

$$B_1 = B_2 = B_3 = \ldots = \bigcup_{n=1}^{\infty} A_n$$

$$\therefore \bigcap_{1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n, \text{ also } C_n = \bigcap_{n}^{\infty} A_k = A_n \qquad \therefore \cup C_n = \cup A_n.$$

So for $A_n \uparrow$, $\limsup A_n = \liminf A_n = \bigcup_{1}^{\infty} A_n$.

Similarly if $A_n \downarrow$ then $B_n = A_n$ $\therefore \lim B_n = \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} A_n$

and $C_n = \bigcap_{k=n}^{\infty} A_k = \bigcap_{k=1}^{\infty} A_k$ $\therefore \bigcup_{1}^{\infty} C_n = \bigcap_{n=1}^{\infty} A_n$

$\therefore \limsup A_n = \liminf A_n = \bigcap_{n=1}^{\infty} A_n$.

Note that every point $w \in \Omega$ may be classified according to the following scheme: (1) w may not belong to any of the subsets A_n. (2) w may belong to only finitely many subsets A_n. (3) w may belong to infinitely many subsets of the sequence of $\{A_n\}$ and may not belong to infinitely many subsets of the sequence

$\{A_n\}$. As an example w may belong to $A_1, A_3, A_5, \ldots, A_{2n+1}, \ldots$ and w may not belong to $A_2, A_4, A_6, \ldots, A_{2n}, \ldots$ (4) w may belong to infinitely many subsets of $\{A_n\}$ and may not belong to only finitely many subsets of $\{A_n\}$.

Now $\limsup A_n$ consists of all those points w which belongs to infinitely many $\{A_n\}$ $\therefore$ it consists of those which belong to category (3) and (4) where as $\liminf A_n$ consists of all those w's which belong to all but a finite number of the sequence of sets $\{A_n\}$ and so belong to category (4). Now if $w \in \liminf A_n$ then $w \in \bigcap_{m=n_0}^{\infty} A_m$ for some finite $n_0 \Rightarrow w$ must belong to infinitely many A_n's. Hence $w \in \limsup A_n$.

Hence $\liminf A_n \subset \limsup A_n$.

Example: Consider two subsets B and C.

Let $A_{2n+1} = B$ for $n = 0, 1, 2, \ldots, \ldots$

$A_{2n} = C$

$$\therefore \bigcup_{m=n}^{\infty} A_m = B \cup C \qquad \therefore \limsup A_m = \cap \bigcup_{m=n}^{\infty} A_m = B \cup C$$

$$\text{Now } \bigcap_{m=n}^{\infty} A_m = B \cap C \qquad \therefore \liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m = B \cap C$$

$\therefore$ in this case $\limsup A_n \neq \liminf A_n$ unless $B = C$.

Now if $B \cap C = \emptyset$ then $\limsup A_n = B \cup C$ and $\liminf A_n = \emptyset$.