ST 778. Fall 2007.

Midterm exam. Dr. Fuentes. .

1. (20 points) If $A_1, A_2, \ldots$ are independent events, and we define $p_n = n^{-1}\sum_{i=1}^{n} P(A_i)$, and $N_n = \sum_{i=1}^{n} I_{A_i}$. Then, show (and justify) that

$$n^{-1}N_n - \overset{P_n}{[\text{illegible}]} \longrightarrow 0,$$

where the convergence is "in probability".

2. (20 points) Assume a sequence $\{X_n\}$ of independent identically distributed simple Random Variables (Bernoulli trials), such that $P(X_n = 1) = p$, $P(X_n = 0) = 1 - p$. We say X_n is a success, when $X_n = 1$. Let S_n be the variable with the number of successes in n trials. Show that (explain what theorem you are using),

$$n^{-1}S_n \longrightarrow p$$

where the convergence is "with probability 1".

3. (35 points) Consider the Real line R and let C_1, C_2 and C_3 be the three classes of subsets of R described below,

$C_1 = \{$ the class of all open intervals with real end points $\}$

$C_2 =\{$the class of all open intervals with rational end points $\}$

$C_3 = \{$the class of all single point sets $\}$,

and consider the class of sets $\mathcal{D}$ of the real line, where

$\mathcal{D} =\{$ A: where A is countable or the complement of A is countable $\}$.

Prove that $\sigma(C_3) = \mathcal{D}$.

Show that $\sigma(C_1) = \sigma(C_2)$ but $\sigma(C_3) \subset \sigma(C_1)$ (this is a PROPER inclusion).

Find a set in $\sigma(C_1)$ which is not in $\sigma(C_3)$.

If λ is the Lebesgue measure on the borel subsets of the real line, then prove that if $A \in \sigma(C_3)$, then $\lambda(A) = 0$ or $+\infty$.

4. (25 points) Let $A_1, A_2, A_3, \ldots$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{A}, \mu)$.

Let m be a fixed positive integer. Define $X(\omega) = \sum_{n=1}^{\infty} 1_{A_n}(\omega)$, where 1_{A_n} is an indicator function. Let B_m denote the set of those points of Ω that belong to A_n for at least m of integers n, i.e. $B_m = \{\omega : X(\omega) = \sum_{n=1}^{\infty} 1_{A_n}(\omega) \geq m\}$.

Show first that B_m is measurable (belongs to σ-field $\mathcal{A}$), and that

$$\mu(B_m) \leq \sum_{1}^{\infty} \mu(A_n)$$

1. If $A_1, A_2, \ldots$ independent

$p_n = \frac{1}{n} \Sigma P(A_i)$

$N_n = \Sigma I_{A_i}$

Show $\frac{1}{n} N_n - p_n \xrightarrow{P} 0$, $\frac{1}{n} \Sigma I_{A_i} - \frac{1}{n} \Sigma P(A_i) \to 0$

$\lim_{n\to\infty} P(|\frac{1}{n} N_n - p_n| > \varepsilon) = 0$

$P(|\frac{1}{n} N_n - p_n| > \varepsilon) \le \frac{E(|\frac{1}{n}N_n - p_n|^2)}{\varepsilon^2}$ by Chebyshev's Inequality

$E(\frac{1}{n} \Sigma I_{A_i}) = \frac{1}{n} \Sigma E(I_{A_i}) = \frac{1}{n} \Sigma P(A_i)$

$\therefore E(\frac{1}{n} N_n) = p_n$

$\therefore E(|\frac{1}{n} N_n - p_n|^2) = Var(\frac{1}{n} N_n)$

$$\therefore \frac{Var(\frac{1}{n} N_n)}{\varepsilon^2} = \frac{Var\, N_n}{n^2 \varepsilon^2} = \frac{\overbrace{\Sigma P(A_i)(1-P(A_i))}^{\to < \infty}}{\underbrace{n^2 \varepsilon^2}_{\to \infty \text{ as } n \to \infty}}$$

$\therefore \lim_{n\to\infty} P(|\frac{1}{n} N_n - p_n| > \varepsilon) \le \lim_{n\to\infty} \frac{\Sigma P(A_i)(1-P(A_i))}{n^2 \varepsilon^2} = 0$

$\therefore \lim_{n\to\infty} P(|\frac{1}{n} N_n - p_n| > \varepsilon) = 0$ $\therefore \frac{1}{n} N_n - p_n \xrightarrow{P} 0$

Excellent!

2. $\{X_n\}$ iid Ber(p), $P(X_n=1)=p$, $P(X_n=0)=1-p$

S_n = # successes in n trials

Show $\frac{1}{n}S_n \xrightarrow{wp1} p$, $P(|\frac{1}{n}S_n - p| > \varepsilon \text{ i.o.}) = 0$

$$E(\tfrac{1}{n}S_n) = \tfrac{1}{n}E(S_n)$$
$$= \tfrac{1}{n}\cdot nE(S_1)$$
$$= \tfrac{1}{n}\cdot np$$
$$= p$$

$$p = E(\tfrac{1}{n}S_n)$$

$\therefore P(\lim_{n\to\infty} \frac{1}{n}S_n = p) = 1$ because as $n\to\infty$ $\frac{1}{n}S_n$ will approach $E(\frac{1}{n}S_n)$ with probability 1

And by Strong Law:

$P(\lim_{n\to\infty} \frac{1}{n}S_n = p) \Rightarrow \frac{1}{n}S_n \xrightarrow{wp1} p$

Excellent!

3. $\mathbb{R}$

$C_1 = \{$ class of all open intervals with real end points $\}$

$C_2 = \{$ class of all open intervals with rational end points $\}$

$C_3 = \{$ class of all single point sets $\}$

~~$D = \{ A :$ where A is countable or A^c is countable $\}$~~

Show $\sigma(C_3) = D$

Let $A \in \sigma(C_3)$

$\therefore$ A is a class of all single points on $\mathbb{R}$

$\therefore$ A is not countable

$\therefore$ A^c is countable $\in D$

Since $A^c \in \sigma(C_3)$ because it is a σ-field then

$\sigma(C_3) = D$

Show $\sigma(C_1) = \sigma(C_2)$

$\rightarrow$: Let $\omega \in \sigma(C_1)$

$\therefore$ ω is in the set of all open intervals with real endpoints

Since all real endpoint intervals are contained in intervals with rational end points then ω is also contained within a rational endpoint interval.

$\therefore \omega \in \sigma(C_2)$

$\therefore \sigma(C_1) \subset \sigma(C_2)$

Because real line is dense

$\leftarrow$: Let $\omega \in \sigma(C_2)$

$\therefore$ ω is in the set of all open intervals with rational endpoints

Since all rational endpoint intervals are contained in intervals with real end points then ω is also contained within a real endpoint interval.

$\therefore \omega \in \sigma(C_1)$

$\therefore \sigma(C_2) \subset \sigma(C_1)$

$\therefore \sigma(C_1) = \sigma(C_2)$

Show $\sigma(C_3) \subset \sigma(C_1)$

Let $\omega \in \sigma(C_3)$

$\therefore$ ω is in the set of all single points

Since any point is contained between two real points, because the real line is dense, then ω is in the set of all intervals with real endpoints

$\therefore \omega \in \sigma(C_1)$

$\therefore \sigma(C_3) \subset \sigma(C_1)$

Find a set in $\sigma(C_1)$ which is not in $\sigma(C_3)$

An interval with real end points will always contain a rational point. So if A is the set of all rational points, it cannot be contained in $\sigma(C_1)$.

Show if $A \in \sigma(C_3)$ then $\lambda(A) = 0, +\infty$

Let A contain one single point $\{a\}$

$\therefore A \in \sigma(C_3)$

$\therefore \lambda(A) = \lambda(\{a\}) = a - a = 0$

Let A contain more than one single point $\{a_i\}$ $i = 1, 2, \ldots$

$\therefore A \in \sigma(C_3)$

$\therefore \lambda(A) = \lambda(\cup(a_i, a_j]_{i \neq j}) = \sum_{i \neq j} a_j - a_i = \infty$

because there are an infinite amount of single points that can belong to A

4. $A_1, A_2, A_3, \ldots$ measurable sets in measure space $(\Omega, \mathcal{A}, \mu)$

$X(\omega) = \sum_{n=1}^{\infty} 1_{A_n}(\omega)$

$B_m = \{ \omega : X(\omega) = \sum_{i=1}^{\infty} 1_{A_n}(\omega) \geq m \}$, $m > 0$ Fixed

Show B_m measurable:

Let $\sum^{\infty} 1_{A_n}(\omega) = m$

$\therefore$ ω belongs to m number of A_n's

Since $A_i \in \mathcal{A} \Rightarrow \omega \in \mathcal{A}$

Let $\sum^{\infty} 1_{A_n}(\omega) > m$ but still Finite

$\therefore$ ω belongs to a finite # of A_n's $> m$

Since $A_i \in \mathcal{A} \Rightarrow \omega \in \mathcal{A}$

Let $\sum^{\infty} 1_{A_n}(\omega) = \infty$

$\therefore$ $\omega \in \limsup A_n \in \mathcal{A}$ $\therefore$ B_m is measurable

Show $\mu(B_m) \leq \sum_1^{\infty} \mu(A_n)$

Let $\omega \in B_m$

$\therefore$ ω belongs to atleast m number of A_n's

$\therefore$ $\omega \in \bigcup_1^{\infty} A_n$

$\therefore$ $\mu(B_m) \leq \mu(\bigcup_1^{\infty} A_n) \leq \sum_{i=1}^{\infty} P(A_n)$ by countable subadditivity

Excellent!