Spectral methods are a powerful tool for studying the spatial-temporal structure of random fields and generally offer significant computational benefits.

Fourier Analysis

Discrete Fourier Analysis: A discrete Fourier analysis of a spatial process, also called a harmonic analysis, is a decomposition of the process into a sum of sinusoidal components (sines and cosines waves). The coefficients of these sinusoidal components are the discrete Fourier transform of the process.

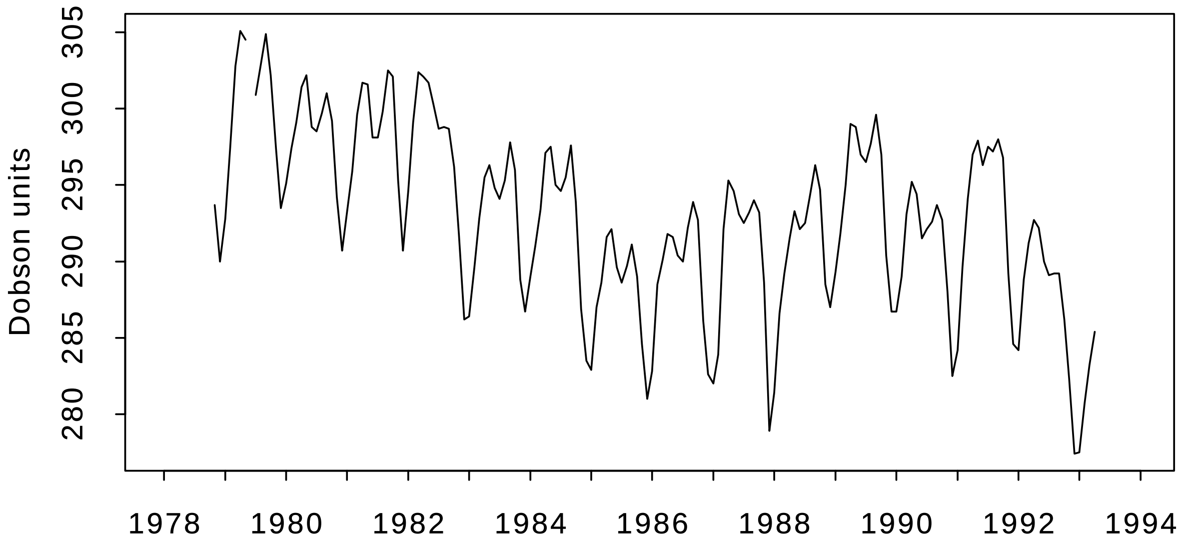


Figure 1: Monthly average total ozone levels (TOMS), 65^oS to 65^oN.

Why Fourier basis?

Sinusoids have some characteristic properties which give them a distinguished role to represent a spatial process Z.

- A sinusoid of frequency ω (# cycles/time unit), or period $1/\omega$ (in time domain units), may be written as

$$Z(t) = R\cos(\omega t + \phi)$$

where R is the amplitude and ϕ is the phase. If the location is changed to $u = (t - a)/b$ which incorporates a change of both origin and scale, $Z(t)$ becomes

$$Z^*(u) = Z(a + ub) = R\cos(\omega ub + \phi + \omega a) = R'\cos(\omega' u + \phi')$$

where $R' = R$, $\omega' = \omega b$ and $\phi' = \phi + \omega a$. Thus the amplitude is unchanged, the frequency is multiplied by b (the reciprocal of the change in the space domain), and the phase is altered by an amount involving the change of space origin and the frequency of the sinusoid. *Since the space origin associated with a set of data is often arbitrary, the simplicity of these relationships is useful.*

- The sum of sinusoids with a common frequency is another sinusoid with the same frequency. In fact, since

$$R\cos(\omega t + \phi) = R\cos(\omega t)\cos(\phi) - R\sin(\omega t)\sin(\phi)$$

any sinusoid with frequency ω is a linear combination of the two basis functions $\cos(\omega t)$ and $\sin(\omega t)$, and the converse is also true.

- A further useful feature of the sinusoids is their behavior under sampling, i.e. when we observe a process that is defined on a continuous space at an equally spaced set of values in a lattice (there is lost of information).

  If the distance between neighboring observations in the lattice is Δ, the sinusoids

  $R\cos(\omega s + \phi)$ and $R\cos(\omega' s + \phi)$ are indistinguishable if $\omega - \omega'$ is a (integer) multiple of $2\pi/\Delta$.

  This phenomenon known as *aliasing*. Aliasing is a relatively simple phenomenon: when one takes a discrete set of observations on a continuous function, information is lost.

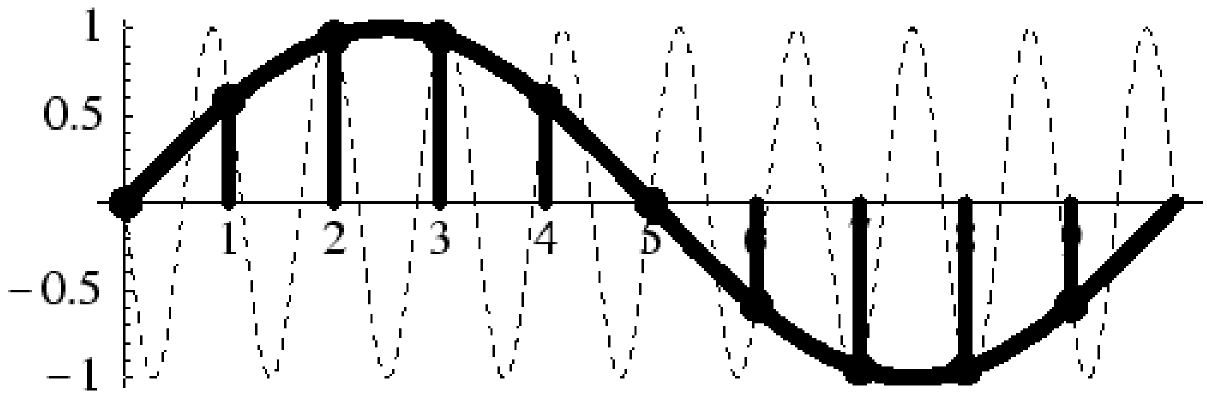


Figure 2: Undersampling in space.

Ozone example: The dominant part of the annual cycle may be expected to be of the form

$$S(t) = \mu + R\cos(2\pi(\omega t + \phi)) = \mu + A_1\cos(2\pi\omega t) + B_1\sin(2\pi\omega t)$$

where the frequency is $\omega = 1/12$ cycles per month.

The seasonal behavior in the ozone data could not be described by a single sinusoid with the annual frequency (1/12 cycle per month). If a semi-annual wave is added to the model the resulting five-parameter model is

$$\mu + A_1\cos(2\pi\omega t) + B_1\sin(2\pi\omega t) + A_2\cos(4\pi\omega t) + B_2\sin(4\pi\omega t)$$

Frequencies that are integer multiples of $1/n$ are said to be harmonic with respect to the span of the data, and are known as the Fourier frequencies. A sinusoid with the jth Fourier frequency, i.e. frequency j/n, executes j complete cycles in the span of the data, thus providing a useful interpretation of the index j .

Fourier series for periodic functions

We could model a time series of length N

$$X(t) = \mu + \sum_{j=1}^{N/2} [A_j \cos(2\pi\omega_j t) + B_j \sin(2\pi\omega_j t)]$$

for $\omega_j = j/N$ where the A_j and B_j rv's are uncorrelated with zero mean and variance σ_j^2, such that $\text{var}(X(t)) = \sigma^2 = \sum_{j=1}^{N/2} \sigma_j^2$. Thus, we decompose the process variance into $N/2$ components, each associated with the expected squared amplitude of sinusoids of a particular frequency.

For mathematical convenience we could use complex exponentials instead of sinusoids directly: $e^{ix} = cosx + isinx$. *Thus,* $cosx = \frac{1}{2}(e^{ix} + e^{-ix})$, *and* $sinx = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Fourier transforms

Periodic functions are represented by a discrete set of frequency components; non-periodic functions involve a continuous range of frequencies. Supposed that $g()$ is a real or complex-valued function, we define

$$f(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} g(\mathbf{s}) \exp\{i\boldsymbol{\omega}^t \mathbf{s}\} d\mathbf{s} \tag{1}$$

The function f in (1) is said to be the Fourier transform of g. Then, g has the representation

$$g(\mathbf{s}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\boldsymbol{\omega}) \exp\{-i\boldsymbol{\omega}^t \mathbf{s}\} d\boldsymbol{\omega}$$

The function g and f are said to be a Fourier transform pair. It is often useful to think of functions and their transforms as occupying two domains. These domains are referred to as the time (or space) and frequency domains respectively.

Spectral representation

Can any process $Z(\mathbf{s})$ be represented using Fourier basis? Yes, if it is stationary.

Z is a stationary process when the mean is constant and $\text{cov}(Z(\mathbf{x}+\mathbf{y}), Z(\mathbf{y})) = C(\mathbf{x})$, where C is the **covariance** function that provides a measure of spatial correlation by describing how sample data are related with distance and direction.

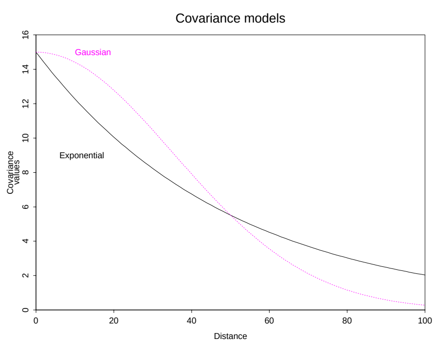


Figure 3: Covariance models: Exponential and Gaussian.

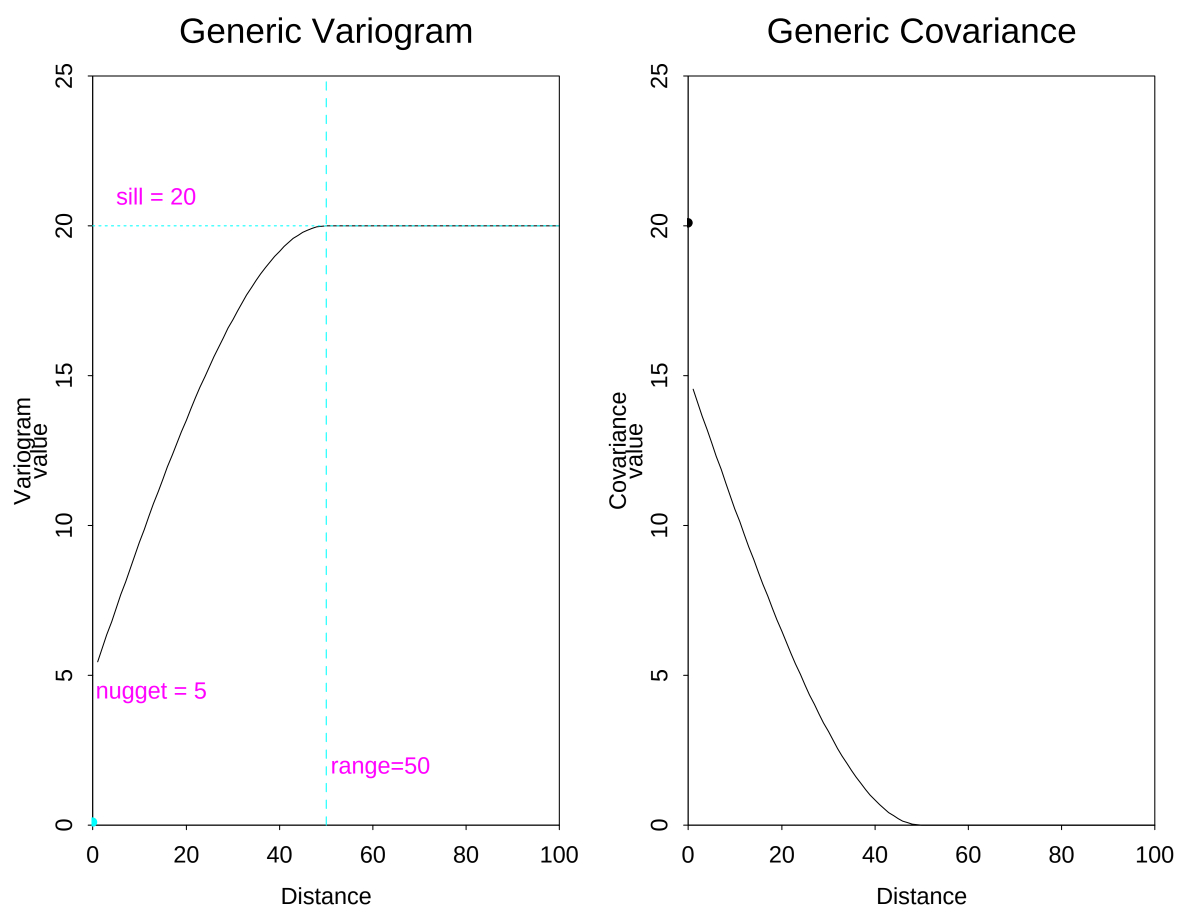


Figure 4: Variogram and Covariance. The variogram is $\gamma(\mathbf{x}) = C(\mathbf{0}) - C(\mathbf{x})$, where C is the covariance function.

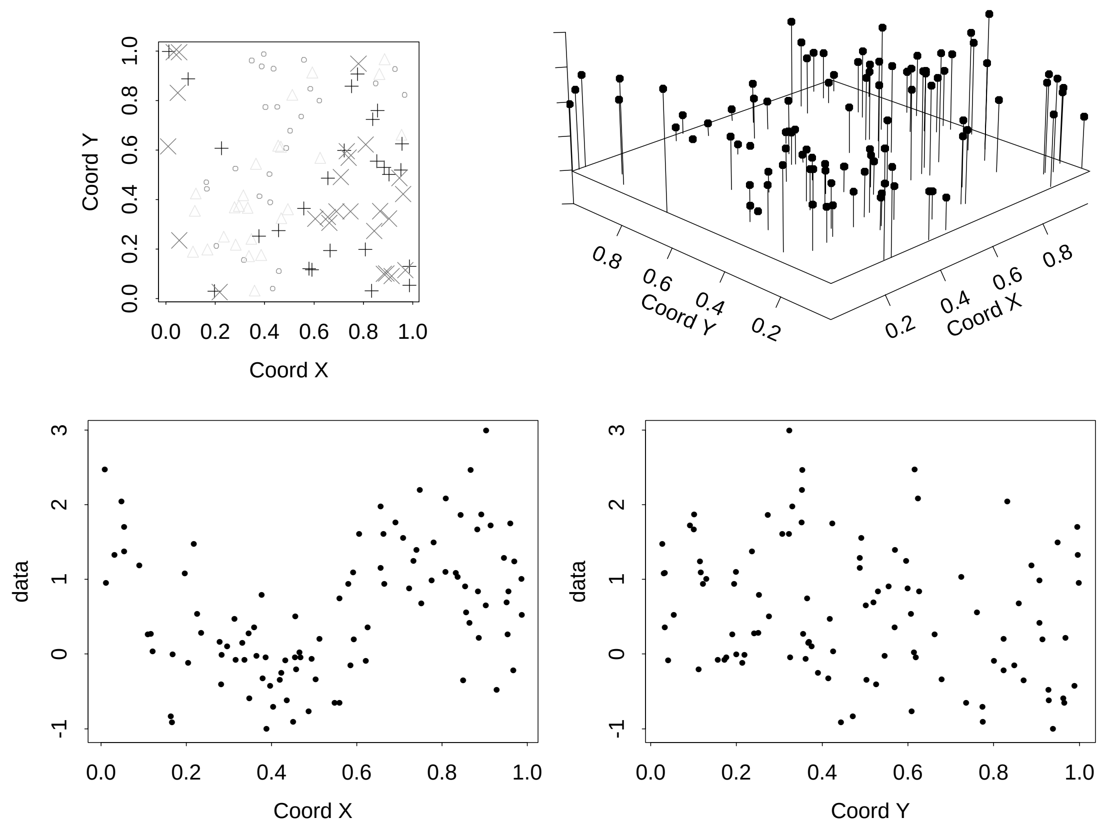


Figure 5: Simulated spatial process using an exponential covariance $C(\mathbf{x}) = \sigma \mathbf{e}^{-|\mathbf{x}|/\rho} + \mathbf{c_0 I}(\mathbf{x} = \mathbf{0})$, with range (ρ) =.3, sill(σ) =1, nugget (c_0)=0. Circles: 1st quantile. Triangles: 2nd. Plus: 3rd. Crosses: 4th.

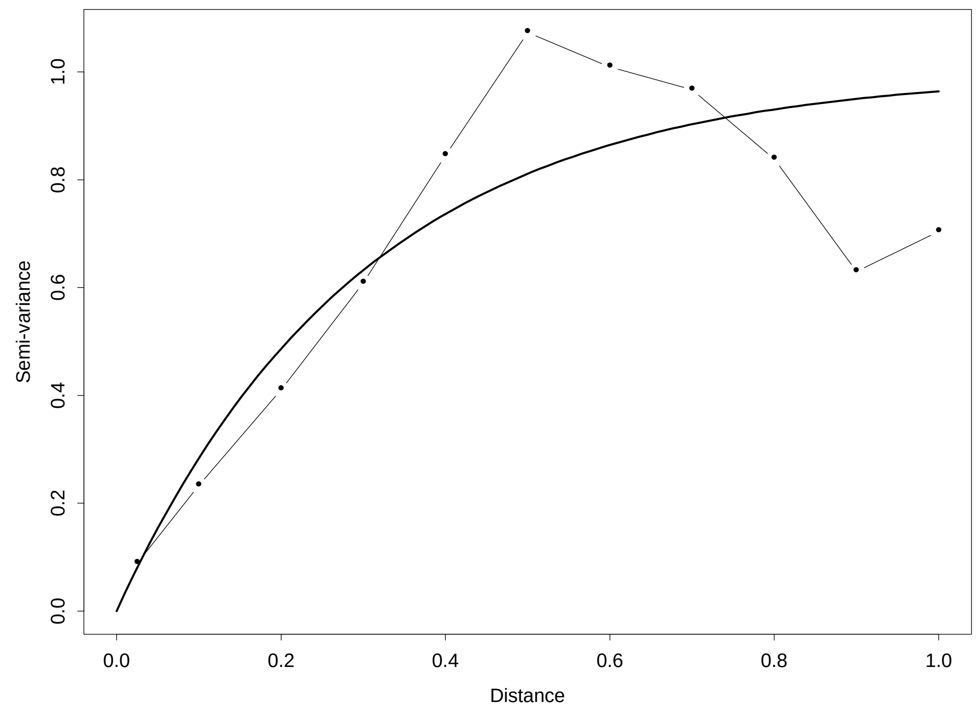


Figure 6: Plot of the theoretical and empirical variogram function. The variogram is $\gamma(\mathbf{x}) = C(\mathbf{0}) - C(\mathbf{x})$, where C is the covariance function. Using an exponential covariance with range $=.3$, sill $=1$, nugget $=0$.

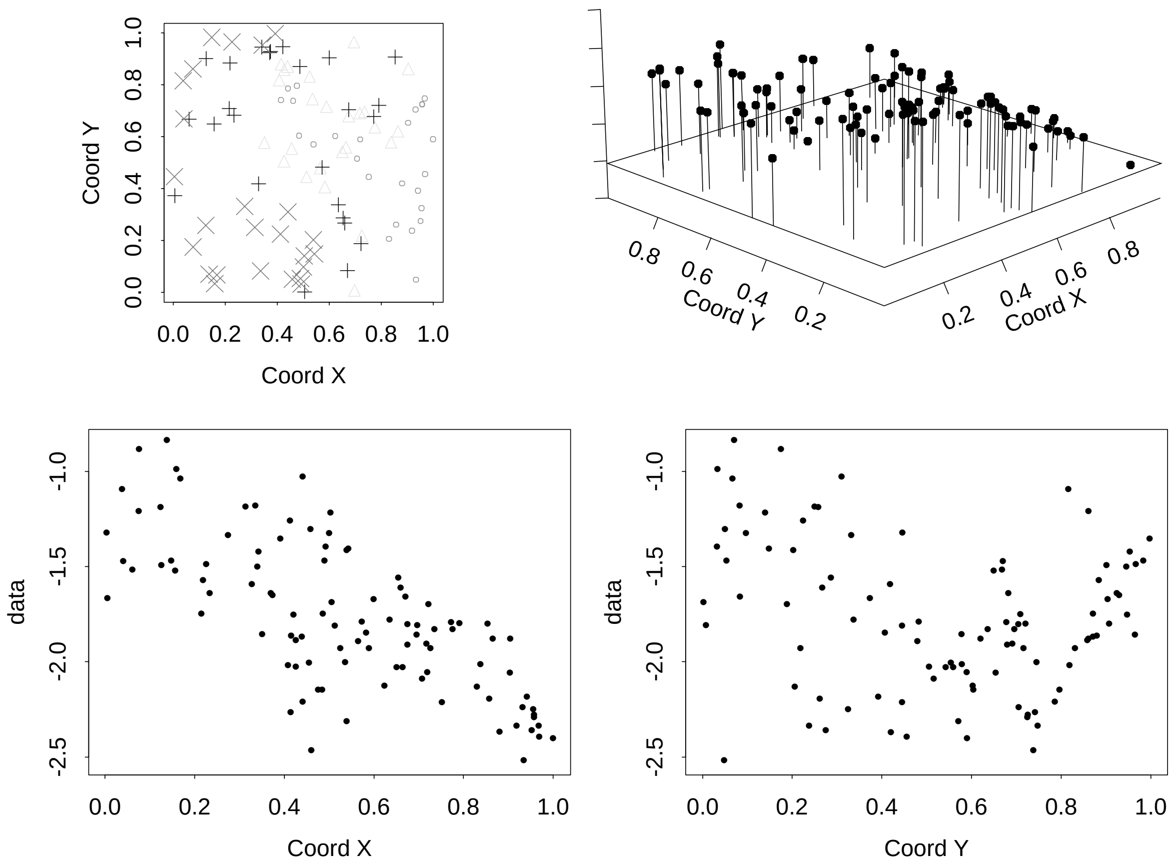


Figure 7: Exponential with range =3, sill =1, nugget =0.

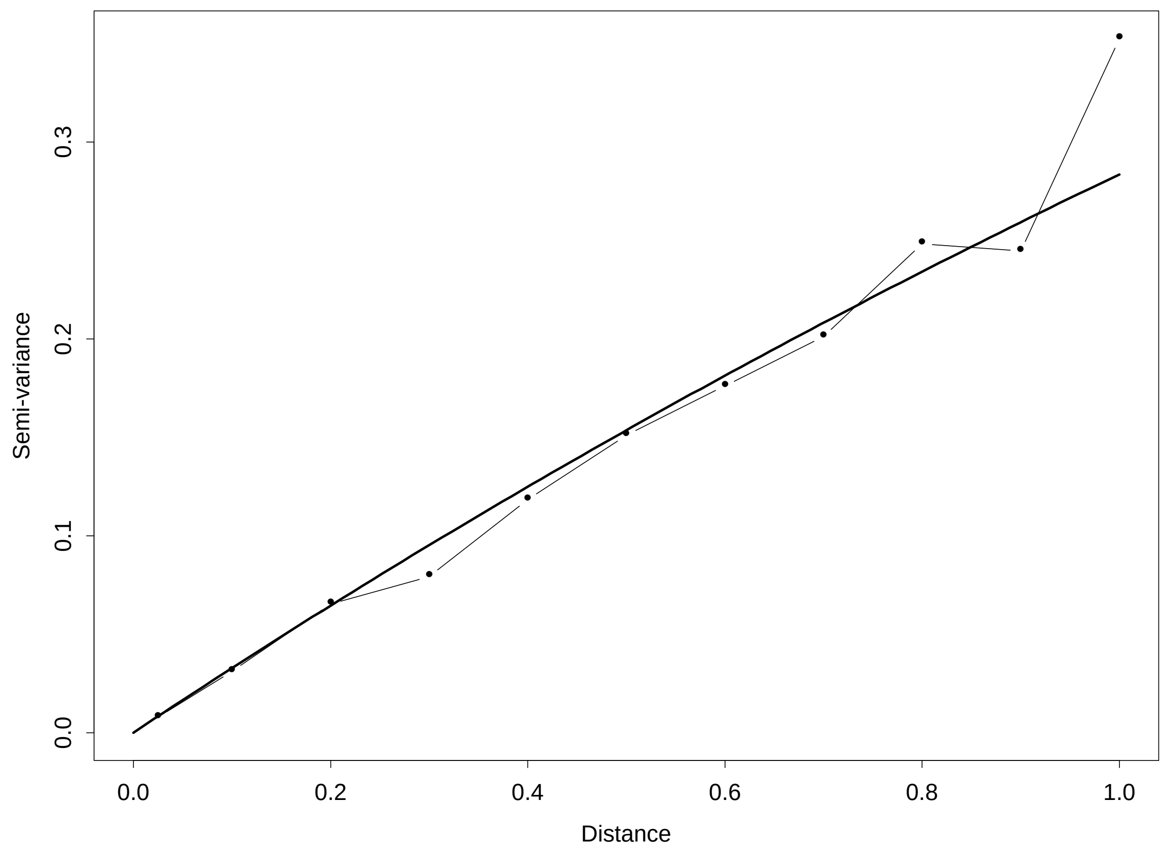


Figure 8: Exponential with range =3, sill =1, nugget =0.

Matérn class of covariances

$C(\mathbf{x})$ is a Matérn stationary covariance at a distance $\mathbf{x}$:

$$C(\mathbf{x}) = \frac{\sigma}{2^{\nu-1}\Gamma(\nu)\alpha^{2\nu}}(2\nu^{1/2}|\mathbf{x}|/\rho)^{\nu}\mathcal{K}_{\nu}(2\nu^{1/2}|\mathbf{x}|/\rho),$$

where $\mathcal{K}_{\nu}$ is a modified Bessel function, d is the dimensionality.
Parameters: smoothing parameter $\nu > 0$, range $\rho > 0$ and sill σ.

For $\nu = \frac{1}{2}$, we get $C(\mathbf{x}) = \sigma \mathbf{e}^{-|\mathbf{x}|/\rho}$.

Critical parameter: ν. The larger ν the smoother the process is, Z will be n times mean square differentiable if and only if $\nu > n$.

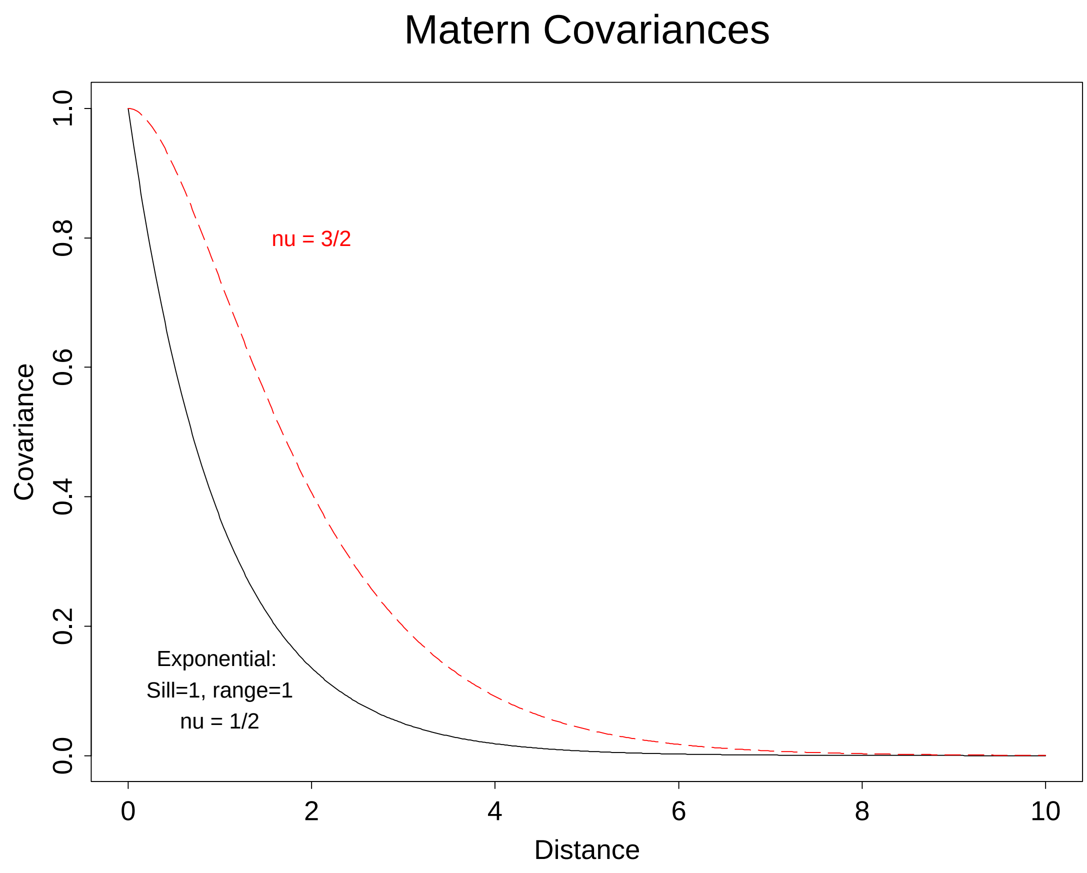


Figure 9: Covariances. Matérn Class for $\nu = 1/2$ (exponential covariance) and $\nu = 3/2$.

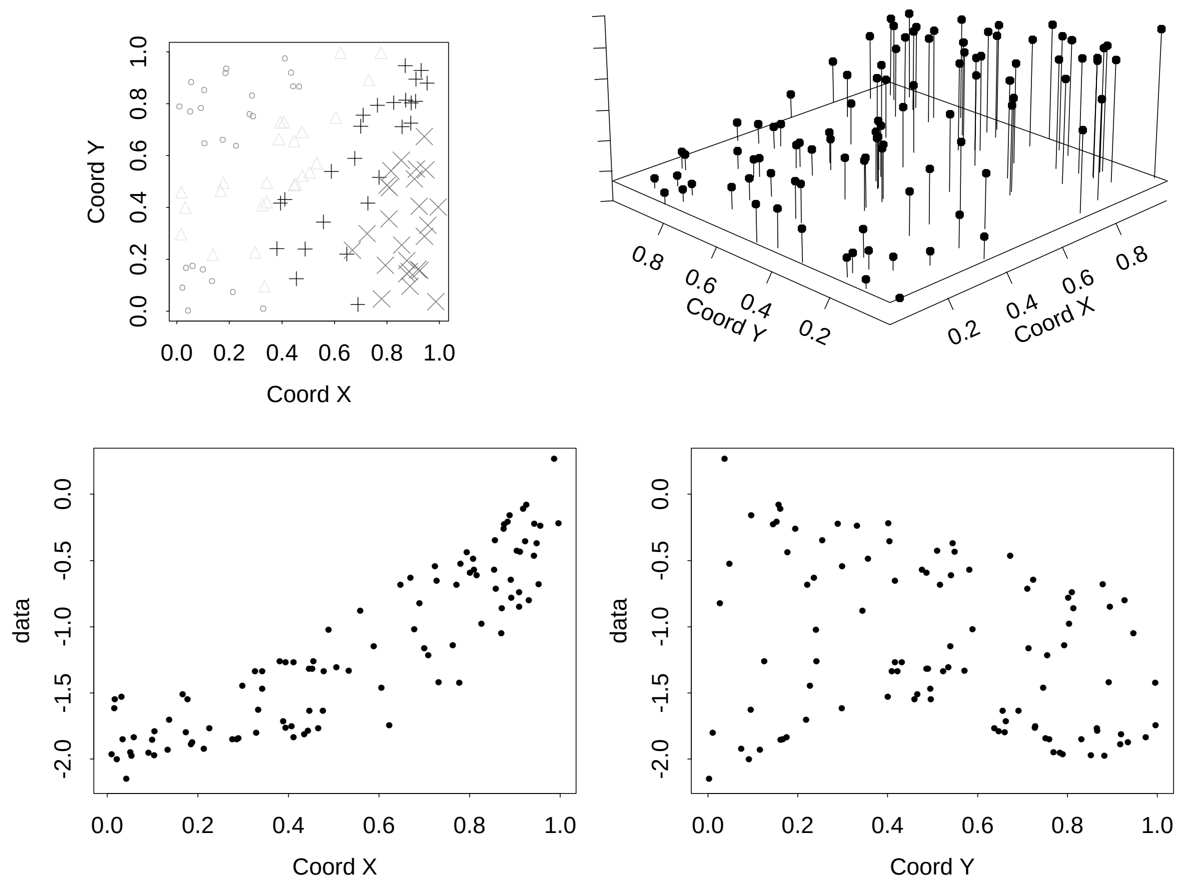


Figure 10: Matérn $\nu = 3$,range =.3, sill =1, nugget =0.

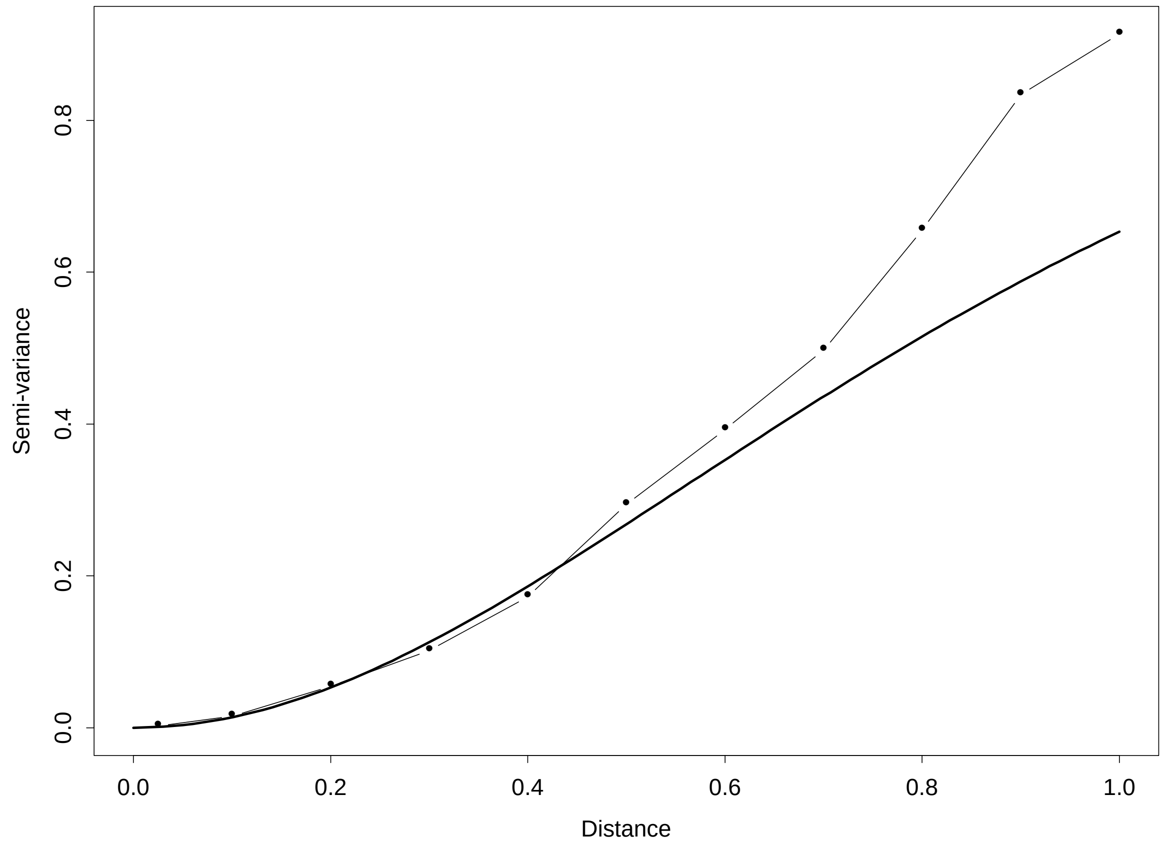


Figure 11: Matérn $\nu = 3$,range =.3, sill =1, nugget =0.

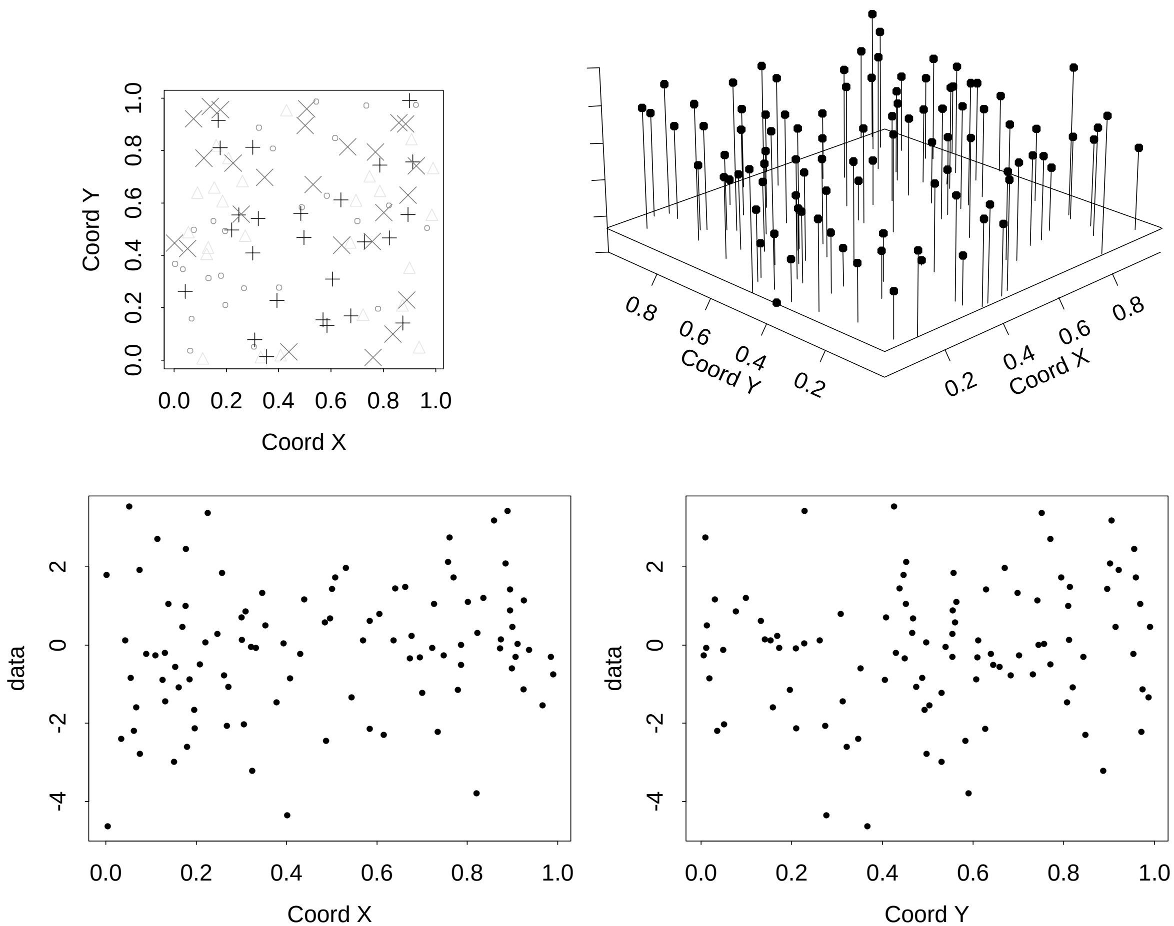


Figure 12: Matérn $\nu = 3$,range =.3, sill =2, nugget =2.

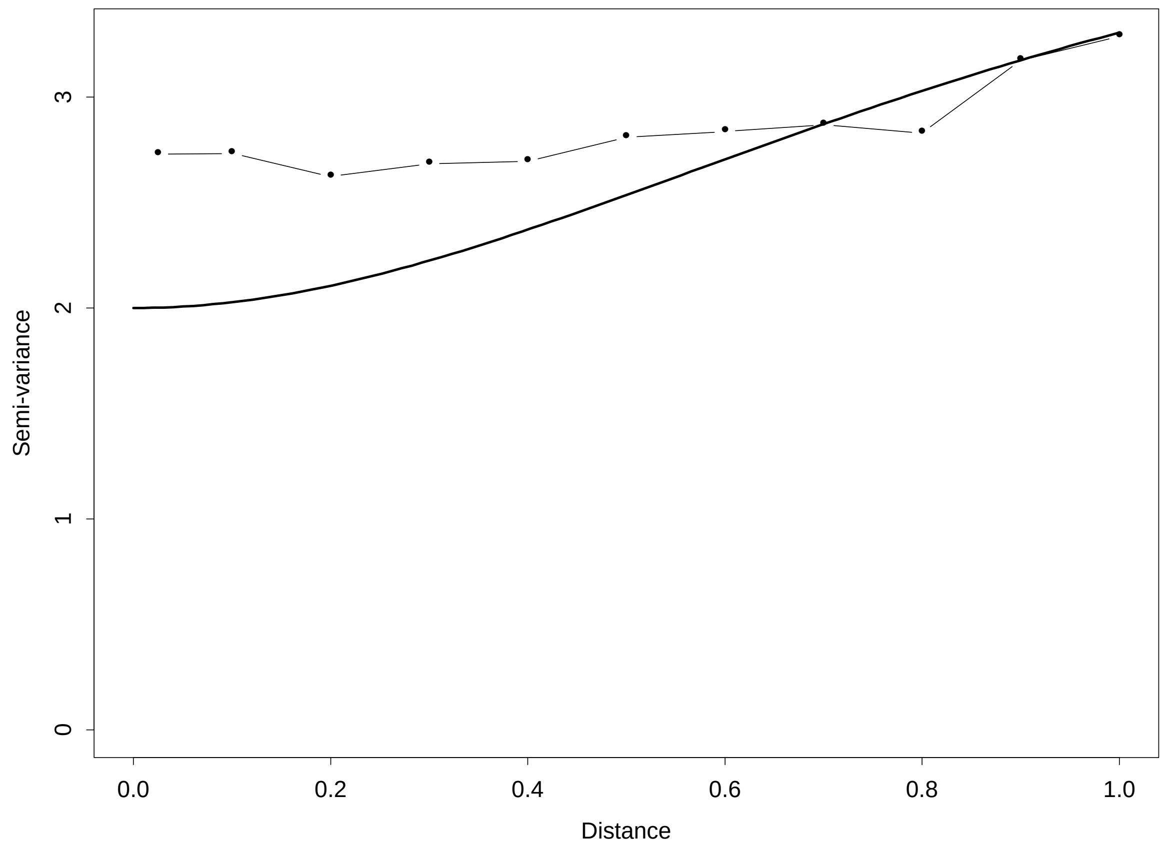


Figure 13: Matérn $\nu = 3$,range =.3, sill =2, nugget =2.

Consider a weakly stationary process Z with mean 0 and covariance C. Before we can apply the ideas of Fourier series and Fourier integrals we must first ask:

- Can we represent a typical realization as a Fourier series? The answer to this question is clearly "No".
- The next question is: Can we represent a typical realization as a Fourier integral? "No". Maybe we can not distribute the power over a continuous range of frequencies, but over a set of frequencies with discontinuities. This will lead to a Fourier type integral called Fourier-Stieltjes.

$$Z(\mathbf{s}) = \int e^{i\mathbf{s}\boldsymbol{\omega}^T} dY(\boldsymbol{\omega}),$$

where Y measures the average contributions from all components with frequencies less than or equal to $\boldsymbol{\omega}$.

The spectral representation theorem

To every stationary $Z(\mathbf{s})$ there can be assigned a process $Y(\boldsymbol{\omega})$ with orthogonal increments, such that we have for each fixed $\mathbf{s}$ the spectral representation:

$$Z(\mathbf{s}) = \int_{\mathbb{R}^2} e^{i\mathbf{s}^T\boldsymbol{\omega}} dY(\boldsymbol{\omega}) \quad (2)$$

The Y process is called the spectral process associated with a stationary process Z. The process Y has orthogonal increments:

$$E[(Y(\boldsymbol{\omega}_3) - Y(\boldsymbol{\omega}_2))(Y(\boldsymbol{\omega}_1) - Y(\boldsymbol{\omega}_0))] = 0,$$

when $(\boldsymbol{\omega}_3, \boldsymbol{\omega}_2)$ and $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_0)$ are disjoint intervals. If we define F as

$$E|dY(\boldsymbol{\omega})|^2] = dF(\boldsymbol{\omega})$$

F is a positive measure.

Bochner's theorem We derive the spectral representation of C :

$$C(\mathbf{s}) = \int_{\mathbb{R}^2} e^{i\mathbf{s}^T \boldsymbol{\omega}} dF(\boldsymbol{\omega})$$

Thus, C is nonnegative definite if and only if it can be represented in the form above where F is real, never-decreasing, and bounded.

If we compare the spectral representation of $C(\mathbf{s})$ and $Z(\mathbf{s})$

$$C(\mathbf{s}) = \int_{\mathbb{R}^2} e^{i\mathbf{s}^T \boldsymbol{\omega}} dF(\boldsymbol{\omega})$$

$$Z(\mathbf{s}) = \int_{\mathbb{R}^2} e^{i\mathbf{s}^T \boldsymbol{\omega}} dY(\boldsymbol{\omega})$$

it will be seen that the elementary harmonic oscillations are respectively $e^{i\mathbf{s}^T \boldsymbol{\omega}} dF(\boldsymbol{\omega})$, $e^{i\mathbf{s}^T \boldsymbol{\omega}} dY(\boldsymbol{\omega})$

We have

$$\text{Sill} = E|Z(\mathbf{s})|^2 = C(\mathbf{0}) = F(\mathbb{R}^2)$$

Thus, F determines the power spectrum of the Z process. We may think of this as a distribution of a spectral mass of total amount $C(\mathbf{0})$ over the $\boldsymbol{\omega}$ axis. F only differs by a multiplicative constant from an ordinary d.f.

If F has a density with respect to Lebesgue measure, this density is the spectral density, $f = F'$, defined as the Fourier transform of the autocovariance function:

$$f(\boldsymbol{\omega}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp(-i\boldsymbol{\omega}^T \mathbf{x}) C(\mathbf{x}) d\mathbf{x},$$

then,

$$\text{Sill} = E|Z(\mathbf{s})|^2 = F(\mathbb{R}^2) = \int_{\mathbb{R}^2} f(\boldsymbol{\omega}) d\boldsymbol{\omega}$$

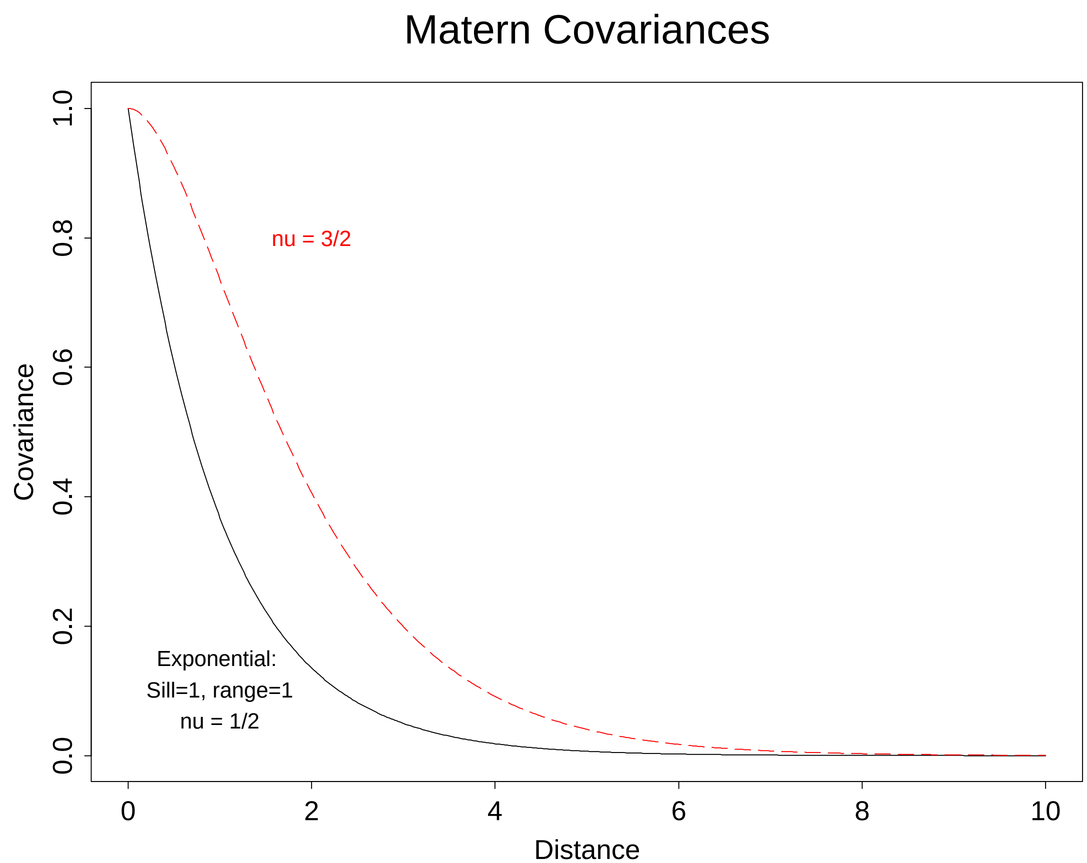


Figure 14: Covariances. Matérn Class for $\nu = 1/2$ (exponential covariance) and $\nu = 3/2$.

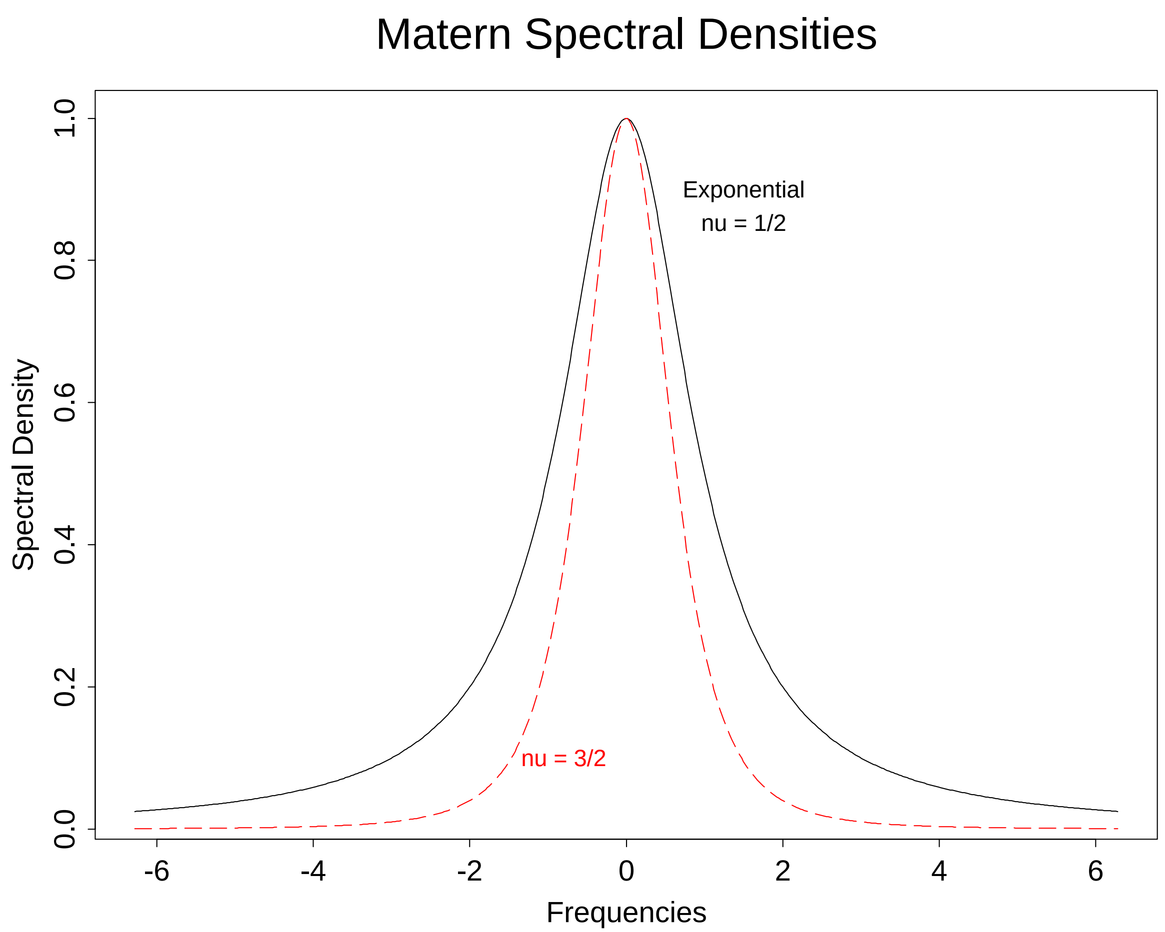


Figure 15: Spectral Densities. Matérn Class for $\nu = 1/2$ (exponential covariance) and $\nu = 3/2$.

Spectral densities

Matérn class

Matérn spectral density:

$$f(\boldsymbol{\omega}) = \phi(\alpha^2 + |\boldsymbol{\omega}|^2)^{\left(-\nu - \frac{d}{2}\right)} \tag{3}$$

with parameters $\nu > 0$, $\alpha > 0$ and $\phi > 0$ (the value d is the dimension of the spatial process Z). Here, the vector of covariance parameters is $\boldsymbol{\theta} = (\phi, \nu, \alpha)$. The parameter α^{-1} can be interpreted as the autocorrelation range. The parameter ν measures the degree of smoothness of the process Z, and ϕ is a scale parameter.

For high frequencies:

$$f(\boldsymbol{\omega}) \sim \phi|\boldsymbol{\omega}|^{(-2\nu - d)} \tag{4}$$

Gaussian model

The density of a a spatial process with an isotropic Gaussian covariance:

$$C(r) = \sigma e^{-\alpha r^2}$$

is

$$f(\omega) = \frac{1}{2}\sigma(\pi\alpha)^{-1/2} e^{-\omega^2/(4\alpha)}$$

Note that C and f both are the same type of exponential functions. The parameter σ is the variance of the process and α^{-1} is a parameter that explains how fast the correlation decays.

For this process the covariance is infinitely differentiable, so the corresponding Z process has mean square derivatives of all orders, then

$$\sum_{j=0}^{m} Z^{(j)}(0)x^j/j! \to Z(x),$$

as m increases, this means that is possible to predict $Z(x)$ perfectly for all x (any location of interest) based on observing $Z(s)$ in a neighborhood of 0, where $x = 0$ is an arbitrary reference point in space. This type of behavior usually would be considered unrealistic for a physical process.

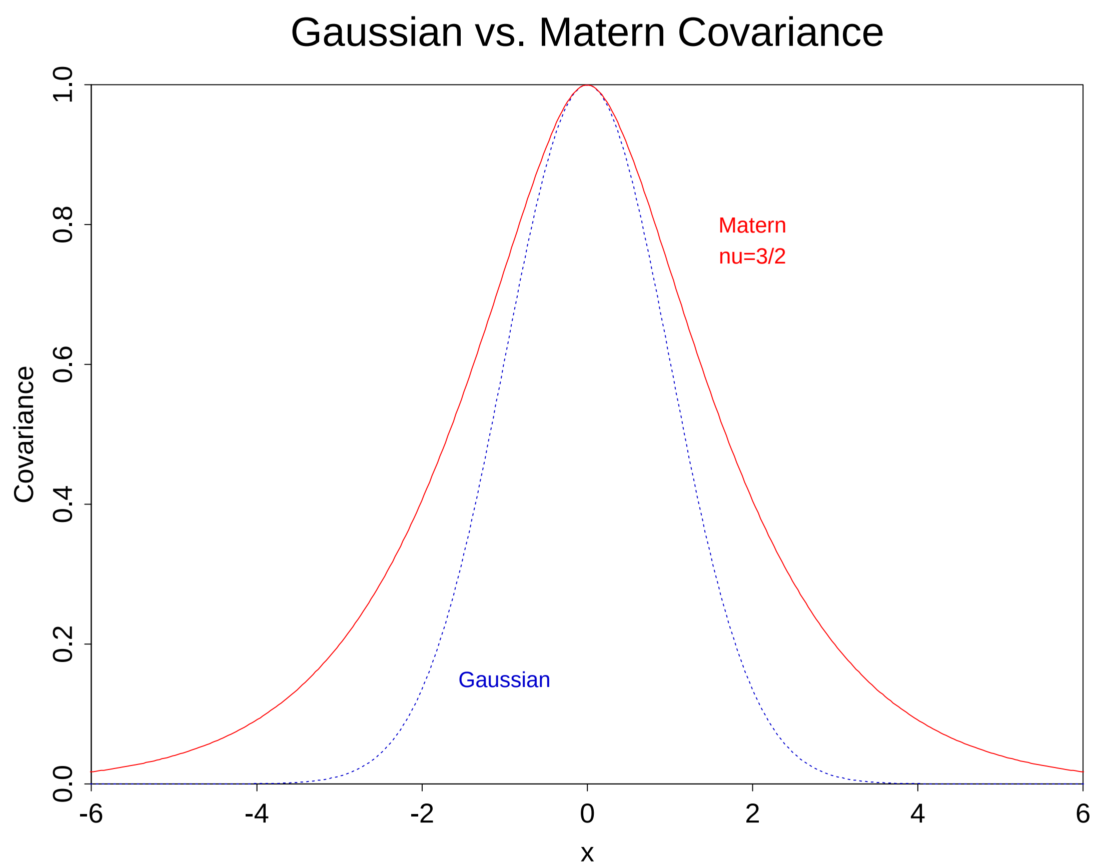


Figure 16: Plot of Gaussian covariance $e^{-x^2/2}$ (dashed line) and matérn covariance $e^{|x|}(1+|x|)$ (solid line) $\nu = 3/2$. Both are of the form $1 - \frac{1}{2}x^2 + O(|x|^3)$ (for small x).

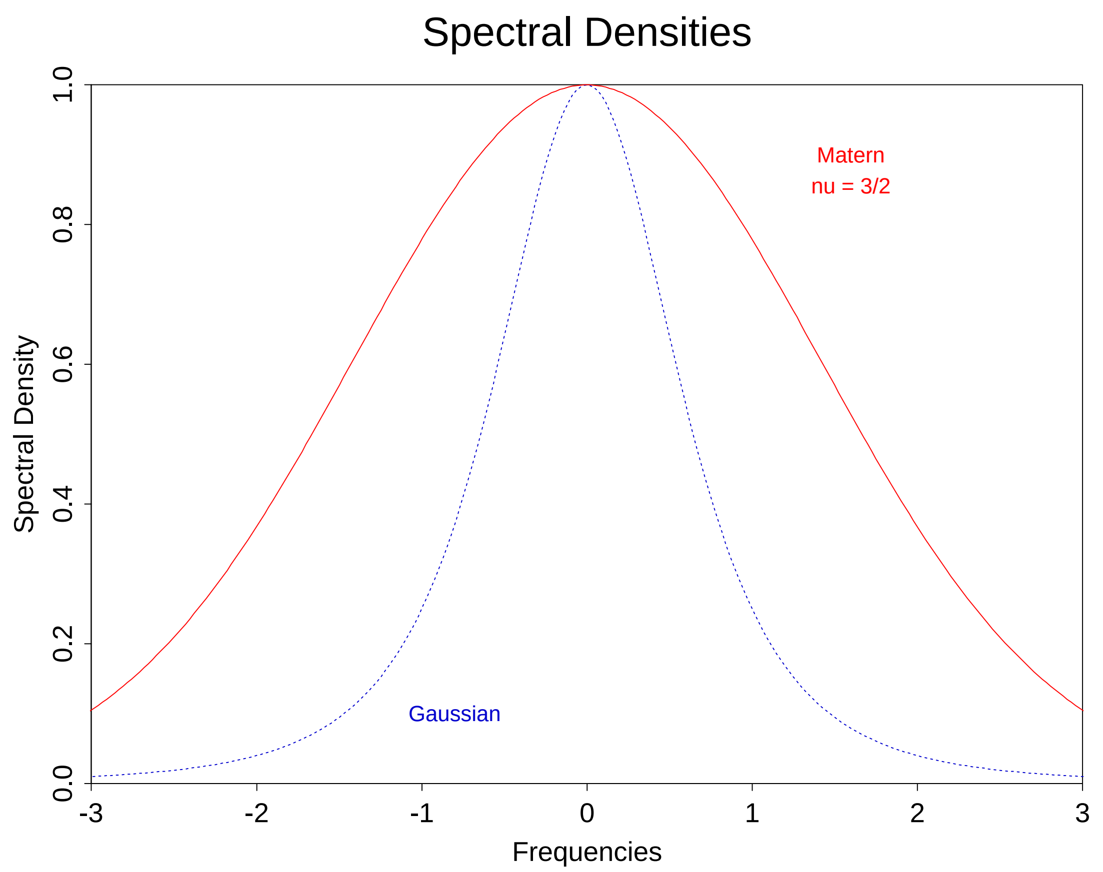


Figure 17: Plot of Gaussian spectral density $e^{-x^2/4}$ (dashed line) and matérn spectral density $(1+|x|^2)^{-\nu+1/2}$ (solid line) $\nu = 3/2$.

Empirical Covariance.

Empirical covariances are likely to be a poor way to distinguish between possible models.

BEHAVIOR AT ORIGIN:

For Gaussian and Matérn $\nu = 3/2$:

$$1 - \frac{1}{2}x^2 + O(|x|^3) \quad \text{(for small x)}$$

However, Matérn $\nu = 3/2$ has only two derivatives at origin and Gaussian is analytic.

More accurate to focus on the high frequency behavior of the spectrum, low frequency behavior of spectrum have little effect on interpolation.

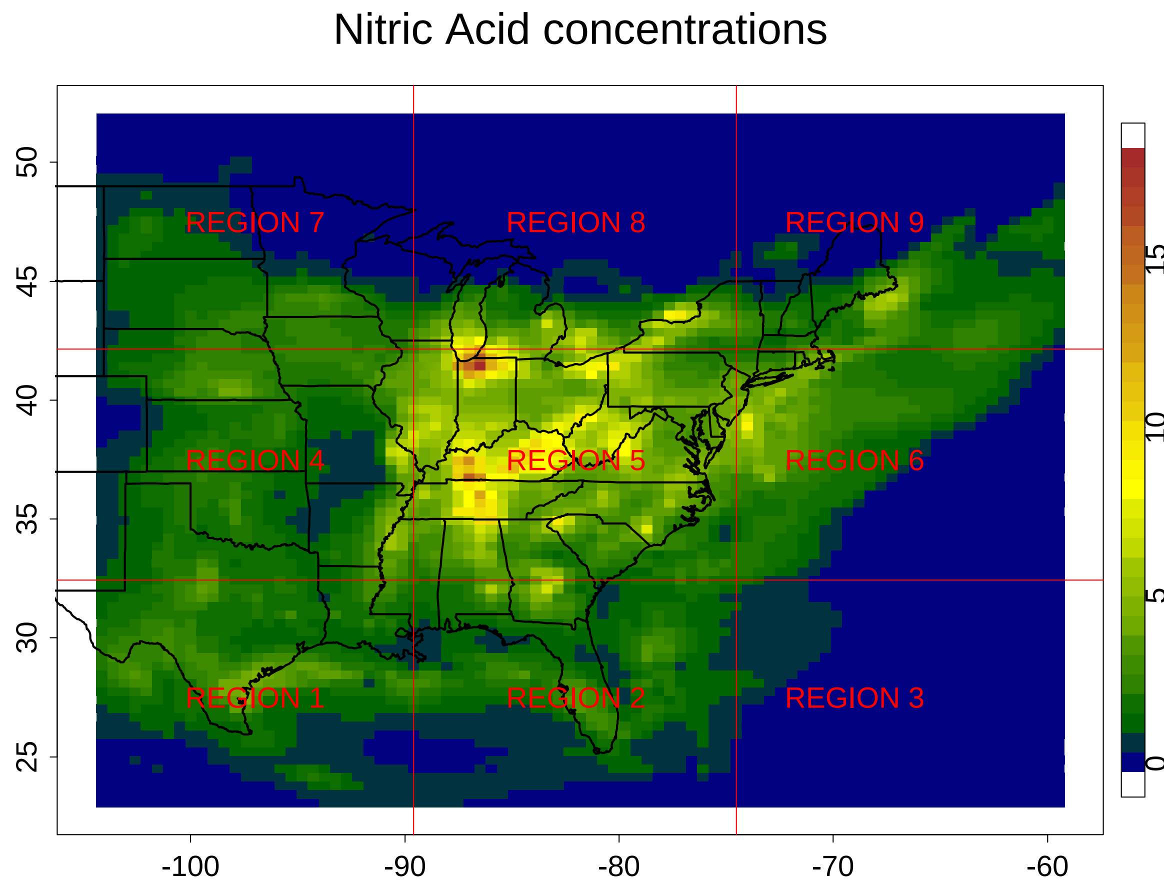


Figure 18: This figure shows the output of Models-3 for the week starting July 11, 1995. We divide the domain in 9 subregions.

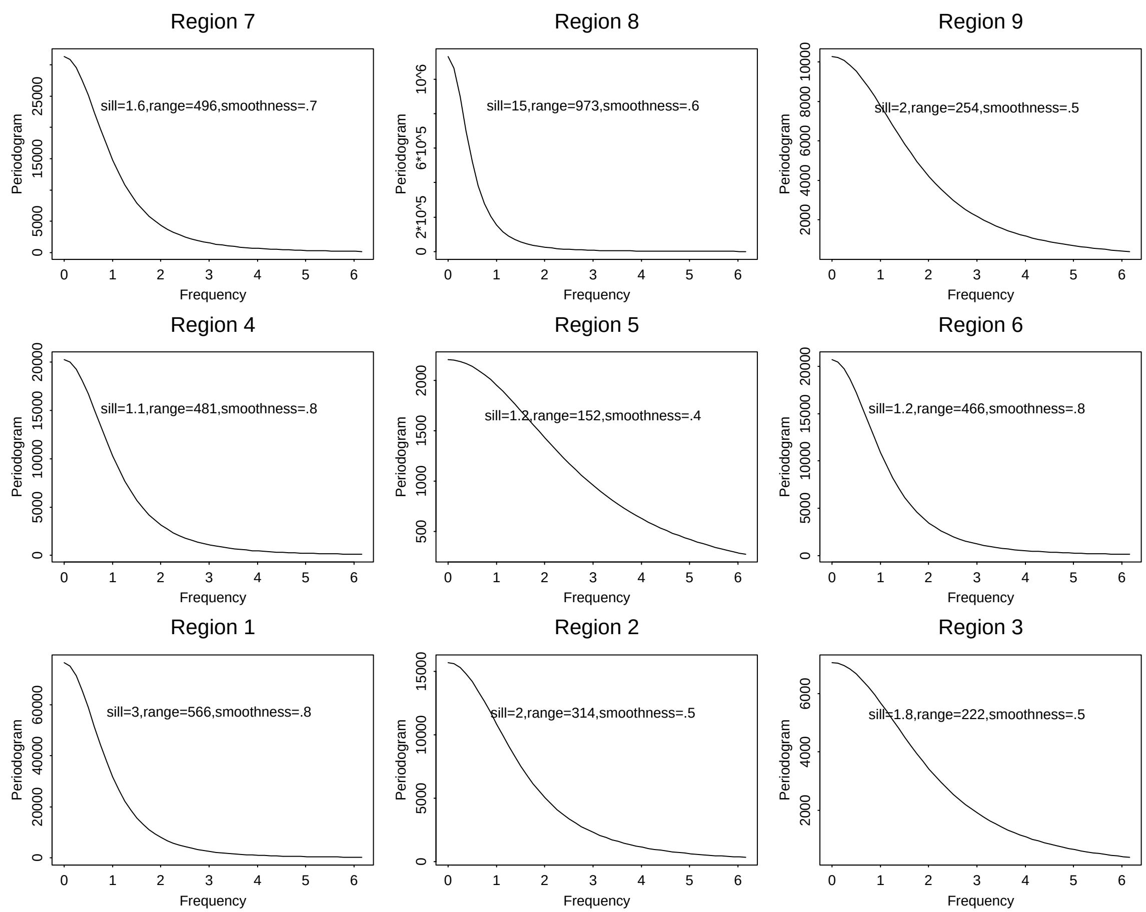


Figure 19: This figure shows the estimated parameters for the Matérn spectral densities of the local stationary processes Z_i for $i = 1, \ldots, 9$.

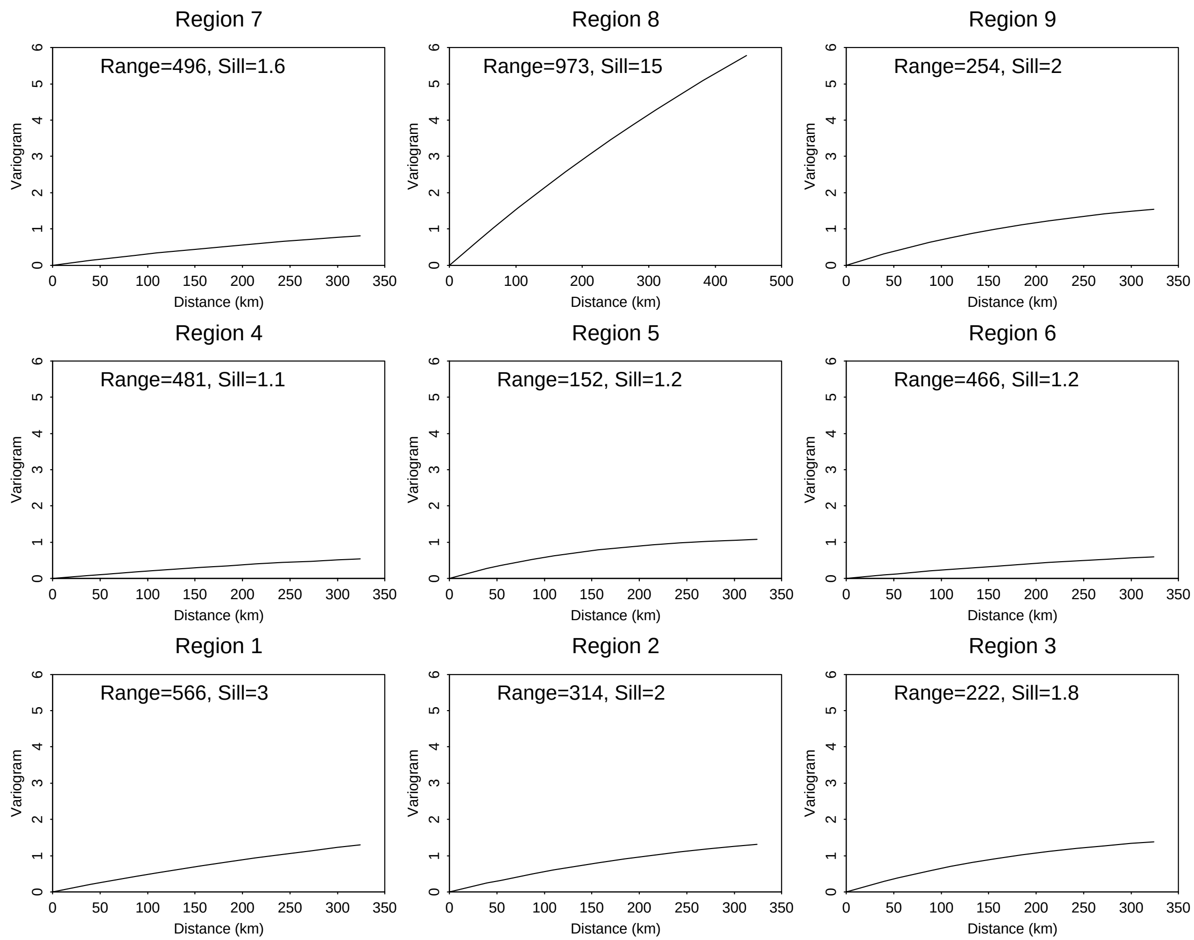


Figure 20: Matérn estimated covariances using a likelihood approach.

ESTIMATING THE SPECTRUM

- The periodogram I_n estimates the spectral density f of a process Z observed in a grid $n \times n$,

$$I_n(\boldsymbol{\omega}) = (2\pi n)^{(-2)} \left| \sum_{\mathbf{j} \in \mathbf{J}} Z(\mathbf{j}) e^{-i\boldsymbol{\omega}^T \mathbf{j}} \right|^2$$

  we consider only the periodogram at the Fourier frequencies $2\pi\mathbf{j}/n$ for $\mathbf{j} \in \mathbf{J} = \{-\lfloor (n-1)/2 \rfloor, \ldots, n - \lfloor n/2 \rfloor\}^2$.

- Periodogram values are approximately independent.

- Periodogram asymptotically unbiased.

The periodogram is simply the discrete fourier transform of the sample covariance.

Spectral weighted-non-linear least squares technique

Consider modeling the spatial structure of Z by fitting a spectral density f to the periodogram values. We could use a weighted non-linear least squares (WNLS) procedure, that gives more weight to higher frequency values because high frequencies are important for interpolation.

We propose using as weights $f(\boldsymbol{\omega})^{-1}$ to give higher weight to higher frequencies. This is reasonable since for large N the approximate standard deviation of the periodogram I_N is $f(\boldsymbol{\omega})$.

This is similar to the weighted least squares method used in the space domain to fit a variogram model (Cressie, 1985). Though, periodogram values are approximately independent while variogram values are not.

Likelihood function

For large datasets calculating the determinants that we have in the likelihood function can be often infeasible. Spectral methods could be used to approximate the likelihood and obtain the MLE of the covariance parameters $\theta_1, \ldots, \theta_r$. Whittle (1954) proposed the following approximation to the Gaussian negative log likelihood:

$$\frac{N}{(2\pi)^2} \sum \{\log f(\boldsymbol{\omega}) + I_N(\boldsymbol{\omega}) (f(\boldsymbol{\omega}))^{-1}\},$$

sum considered at the Fourier frequencies $\boldsymbol{\omega} = (2\pi \mathbf{f}/\mathbf{n})$. We recommend to leave out of the sum the small frequencies, at least the frequency 0, to avoid the problem of estimating the unknown mean of Z. We taper the data giving less weight to boundary observations (to get efficient estimates).

This approximation requires only $O(N log_2 N)$ operations. Simulation studies showed that N needs to be at least 100 to get good estimated MLE parameters.

The spectral MLE estimates are **consistent** in only in one dimension, unless we do tapering or the unbiased version of the sample covariance is used to obtain the periodogram.

The asymptotic covariance matrix (as $N \to \infty$) of the MLE estimates of $\theta_1, \ldots, \theta_r$ is

$$\left\{ \frac{2}{N} \left[\frac{1}{4\pi^2} \int_{[-\pi,\pi]} \int_{[-\pi,\pi]} \frac{\delta \log f(\omega)}{\delta \theta_j} \frac{\delta \log f(\omega)}{\delta \theta_k} d\omega \right]^{-1} \right\}_{jk}$$

This is much easier to compute than the inverse of the Fisher information matrix.

NONSTATIONARY MODELS

- The most extensively studied method for nonstationarity is the *deformation approach* due to Sampson and Guttorp (1992).
- Haas (1995) proposed an approach to nonstationary spatial kriging based on *moving windows*.
- Higdon, Swall and Kern (1999) use a *moving average* specification of a Gaussian process.
- Nychka and Saltzman (1998) present an extension of the *" empirical orthogonal functions"* (EOF) approach that is popular among atmospheric scientists.
- Wikle et al. (2001) use a *wavelet-based* representation for the covariance.
- Fuentes (2001, 2002), and Fuentes and Smith (2001) introduced a nonstationary covariance model with parameteres varying with location.

New Model for nonstationarity

SPATIAL SPECTRA

(Fuentes, 2002)

We present an approach for the spectral analysis of non-stationary spatial processes, $Z(\mathbf{x})$, which is based on the concept of spatial spectra, this means spectral functions which are space dependent, $f_{\mathbf{x}}(\boldsymbol{\omega})$.

The spectral representation of $Z(\mathbf{x})$ is always interpreted as its representation in the form of superposition of sine and cosine waves of different frequencies $\boldsymbol{\omega}$

$$Z(\mathbf{x}) = \int_{\mathbb{R}^2} \exp(i\mathbf{x}^T\boldsymbol{\omega})\phi_{\mathbf{x}}(\boldsymbol{\omega})dY(\boldsymbol{\omega})$$

The functions $\phi_{\mathbf{x}}(\boldsymbol{\omega})$ are slowly varying functions of $\mathbf{x}$ satisfying

$$\int_{\mathbb{R}^2} |\phi_{\mathbf{x}}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega} < \infty.$$

The covariance function, C, of $Z(\mathbf{x})$ is

$$\mathrm{cov}\{Z(\mathbf{x}), Z(\mathbf{y})\} = C(\mathbf{x}_1, \mathbf{x}_2)$$

$$= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \exp\{i(\mathbf{x}_1 - \mathbf{x}_2)^T \boldsymbol{\omega})\} \phi_{\mathbf{x}_1}(\boldsymbol{\omega}) \phi^c_{\mathbf{x}_2}(\boldsymbol{\omega}) d\boldsymbol{\omega}.$$

In particular:

$$\mathrm{var}\{Z(\mathbf{x})\} = C(\mathbf{x}, \mathbf{x}) = \int_{\mathbb{R}^2} |\phi_{\mathbf{x}}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}.$$

Then,

$$f_{\mathbf{x}}(\boldsymbol{\omega}) = |\phi_{\mathbf{x}}(\boldsymbol{\omega})|^2$$

is the spatial spectral density of Z.

Nonparametric spatial spectral estimate

We propose a nonparametric estimate of the spectral density. We first define $J_{\mathbf{x}}(\boldsymbol{\omega}_0)$,

$$J_{\mathbf{x}}(\boldsymbol{\omega}_0) = \Delta \sum_{u_1=x_1-n_1}^{x_1} \sum_{u_2=x_2-n_2}^{x_2} g(\Delta\mathbf{u}) Z(\Delta(\mathbf{x}-\mathbf{u}))$$
$$\exp\{-i\Delta(\mathbf{x}-\mathbf{u})^T \boldsymbol{\omega}_0\},$$

where $\mathbf{s} = (s_1, s_2)$, and $\{g(\mathbf{u})\}$ is a filter. We refer to $|J_{\mathbf{x}}(\boldsymbol{\omega})|^2$ as the spatial periodogram at a location $\mathbf{x}$ for a frequency $\boldsymbol{\omega}$.

$|J_{\mathbf{x}}(\boldsymbol{\omega})|^2$ is an approximately unbiased estimate of $f_{\mathbf{x}}(\boldsymbol{\omega})$ (Fuentes, 2002) but as its variance may be shown to be independent of N it will not be a very useful estimate in practice. Then, we estimate $f_{\mathbf{x}}(\boldsymbol{\omega})$ by "smoothing" the values of $|J_{\mathbf{x}}(\boldsymbol{\omega})|^2$ over neighboring values of $\mathbf{x}$.

More precisely, let W_ρ be a weight function or "window", depending on the parameter ρ. Then, we estimate $f_{\mathbf{x}}(\boldsymbol{\omega}_0)$ by

$$\hat{f}_{\mathbf{x}}(\boldsymbol{\omega}_0) = \sum_{v_1=x_1-n_1}^{x_1} \sum_{v_2=x_2-n_2}^{x_2} W_\rho(\mathbf{v}) \left|J_{\mathbf{x}-\mathbf{v}}(\boldsymbol{\omega}_0)\right|^2$$

Thus, $\hat{f}_{\mathbf{x}}(\boldsymbol{\omega}_0)$ can be interpreted as an average of the total energy of the process contained within a band of frequencies in the region of $\boldsymbol{\omega}_0$ and an region in space in the neighborhood of $\mathbf{x}$.

Parametric estimate

We assume a parametric model for the spectral density, a Matérn with parameter $\boldsymbol{\theta}_i$ changing with location.

$$f_i(\boldsymbol{\omega}) = \phi_i(\alpha_i^2 + |\boldsymbol{\omega}|^2)^{\left(-\nu_i - \frac{d}{2}\right)},$$

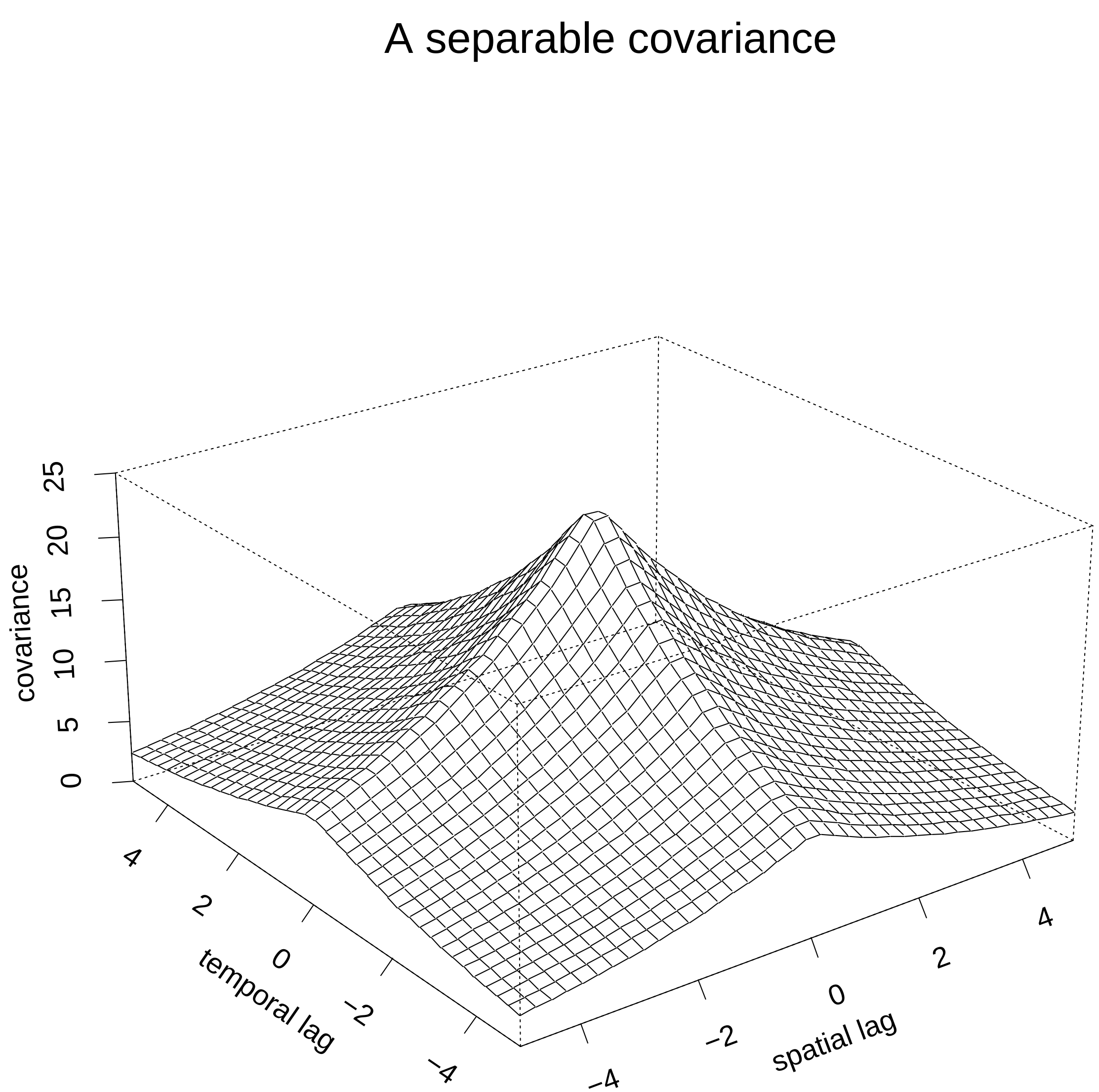
The parameters α_i, ν_i and ϕ_i vary with location.

New Spatial-temporal models

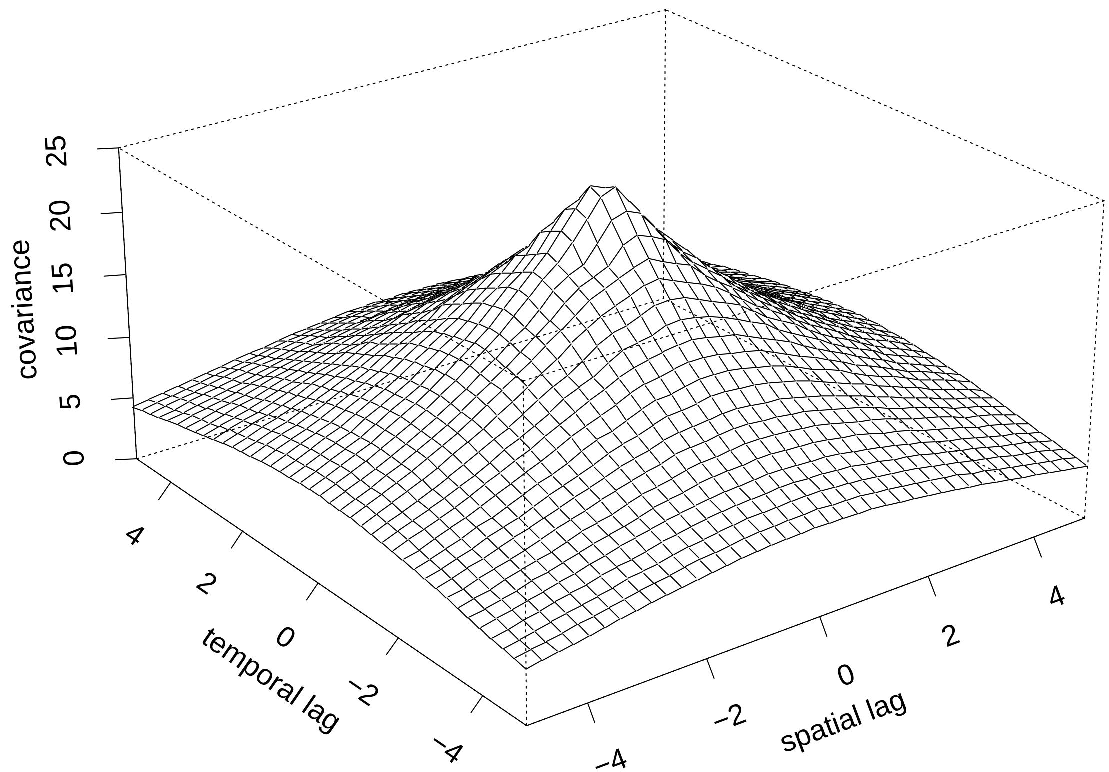
A spatial-temporal field $Z(\mathbf{s}, t)$, where $\mathbf{s}$ represent space and t time, is separable if $\text{Cov}\{Z(\mathbf{s}, t), Z(\mathbf{s}', t')\} = C_1(\mathbf{s}, \mathbf{s}')C_2(t, t')$ for some spatial covariance C_1 and temporal covariance C_2.

A class of nonseparable spatial-temporal models were proposed by Cressie and Huang (JASA, 99), Gneiting (JASA, 02), and by Stein (2003).

Here we define a new class of generalized spectral representations for nonstationary and nonseparable spatial temporal processes. For this new class of spatial temporal models the spectral representation itself and the corresponding spectral distribution function (or spectral density) can change slowly on space and time.


A separable covariance
covariance
0
5
10
15
20
25
4
2
0
−2
−4
temporal lag
−4
−2
0
2
4
spatial lag

A non−separable covariance



Let Z be a nonseparable process, we propose the following representation

$$Z(\mathbf{x}, t) = \int_{\mathbb{R}^3} \exp(i\mathbf{x}^T\boldsymbol{\omega} + it\tau)\phi_{\mathbf{x},\mathbf{t}}(\boldsymbol{\omega}, \tau)dY(\boldsymbol{\omega}, \tau). \tag{6}$$

It is easy to see that then the covariance function, C, of the nonseparable process $Z(\mathbf{x}, t)$ is given by the formula

$$\text{cov}\{Z(\mathbf{x}_1, t_1), Z(\mathbf{x}_2, t_2)\} = C(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2)$$

$$= \int_{\mathbb{R}^3} \exp\{i(\mathbf{x}_1 - \mathbf{x}_2)^T\boldsymbol{\omega})\}\exp\{i(t_1 - t_2)^T\tau\}\phi_{\mathbf{x}_1,t_1}(\boldsymbol{\omega}, \tau)\phi^c_{\mathbf{x}_2,t_2}(\boldsymbol{\omega}, \tau)d\boldsymbol{\omega}d\tau \tag{7}$$

In particular

$$\text{var}\{Z(\mathbf{x}, t)\} = C(\mathbf{x}, t; , \mathbf{x}, t) = \int_{\mathbb{R}^2} |\phi_{\mathbf{x},\mathbf{t}}(\boldsymbol{\omega}, \tau)|^2 d\boldsymbol{\omega}d\tau \tag{8}$$

Locally (in a neighborhood of $\mathbf{s}_i = (\mathbf{x}_i, t_i)$) we propose the following parametric model for ϕ,

$$\phi_{\mathbf{s}_i}(\boldsymbol{\omega}, \tau) = \gamma_i(\alpha_i^2 + \|\boldsymbol{\omega}\|^2 + \beta_i^2|\tau|^2)^{(-\nu_i - \frac{d}{2})/2} \tag{9}$$

with parameters $\nu_i > 0$, $\gamma_i > 0$ and $\gamma_i > 0$, where d is the dimension ($d = 3$ for space and time). The parameter $1/\alpha_i$ explains the rate of decay of the spatial dependency; for the temporal component the rate of decay is explained by β_i/α_i; γ_i is a scale parameter.

Conclusions

- A Stationary spatial process can be always represented in terms of sines and cosines (Fourier basis). This is called a spectral representation.
- Spectral methods offer enormous computational benefits (Whittle likelihood, FFT).
- A nonstationary process can be also represented using Fourier basis with varying amplitude (Fuentes, 02). Wavelets is an alternative approach used by Wikle, and Nychka amount others.
- Nonseparable (valid) spatial-temporal covariance models can be easily obtained using spectral methods (Cressie and Huang, 99 , Stein, 03).
- Nonstationary and nonseparable (valid) spatial-temporal covariance models can be also obtained using spectral methods (Fuentes, 03).

Books about spectral methods for spatial data:

- Christakos, G. (1992).
- Stein (1999).
- Yaglom (1987).

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