Ch.2-Ex.28: (a)  

i. Here, prior distribution of $\theta \sim N(0, 1)$. Hence,

\[
p(\theta | y) \propto p(\theta | y)p(\theta) = \left(\frac{1}{\sqrt{2\pi}}\right) \exp \left\{ -\frac{(y - \theta)^2}{2} \right\} \left(\frac{1}{\sqrt{2\pi}}\right) \exp \left\{ -\frac{\theta^2}{2} \right\} \\
\propto \exp \left\{ -\frac{(y - \theta)^2}{2} \right\} \exp \left\{ -\frac{\theta^2}{2} \right\} \\
= \exp \left\{ -\frac{(\theta^2 - y\theta + \frac{y^2}{2})}{2 \times \frac{1}{2}} \right\} \\
\propto \exp \left\{ -\frac{(\theta - \frac{y}{2})^2}{2 \times \frac{1}{2}} \right\}
\]

If $Z \sim N(\mu, \sigma)$, then $p(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\} \propto \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\}$. Here

\[
p(\theta | y) \propto \exp \left\{ -\frac{(\theta - \frac{y}{2})^2}{2 \times \frac{1}{2}} \right\}, \text{ So}
\]

\[
p(\theta | y) = \frac{1}{\sqrt{\pi}} \exp \left\{ -(\theta - \frac{y}{2})^2 \right\}
\]

ii. Refer to part (i), $\theta | y \sim N(\frac{y}{2}, \sqrt{\frac{1}{2}})$ is a normal distribution.

iii. Mean is $\frac{y}{2}$, SD is $\sqrt{\frac{1}{2}}$. 


(b)  i. Here, prior distribution of \( \theta \sim N(m, \sigma) \). Hence,

\[
p(\theta | y) \propto p(\theta | y) p(\theta) = \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-(y - \theta)^2}{2\sigma_y^2} \right\} \cdot \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-(\theta - m)^2}{2\sigma^2} \right\}
\]

\[
\propto \exp \left\{ -\frac{(y - \theta)^2}{2\sigma_y^2} \right\} \exp \left\{ -\frac{(\theta - m)^2}{2\sigma^2} \right\} \exp \left\{ -\frac{(y - \theta)^2\sigma^2 - (\theta - m)^2\sigma_y^2}{2\sigma_y^2\sigma_y^2} \right\} \exp \left\{ -\frac{(\sigma^2 + \sigma_y^2)\theta^2 - 2(y\sigma^2 + m\sigma_y^2)\theta}{2\sigma_y^2\sigma_y^2} \right\}
\]

\[
= \exp \left\{ -\frac{\theta^2 - 2y\sigma^2 + m\sigma_y^2}{\sigma^2 + \sigma_y^2} \right\}
\]

\[
\propto \exp \left\{ -\frac{(\theta - y\sigma^2 + m\sigma_y^2)}{\sigma^2 + \sigma_y^2} \right\}
\]

If \( Z \sim N(\mu, \sigma) \), then \( p(z) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\} \propto \exp \left\{ -\frac{(z - \mu)^2}{2\sigma^2} \right\} \). Here

\[
p(\theta | y) \propto \exp \left\{ -\frac{(\theta - y\sigma^2 + m\sigma_y^2)}{\sigma^2 + \sigma_y^2} \right\}
\]

So

\[
p(\theta | y) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{(\theta - y\sigma^2 + m\sigma_y^2)^2}{2(\sigma^2 + \sigma_y^2)} \right\} \exp \left\{ -\frac{(\sigma^2 + \sigma_y^2)\theta^2 - 2(y\sigma^2 + m\sigma_y^2)\theta}{2(\sigma^2 + \sigma_y^2)} \right\}
\]

ii. Refer to part (i), \( \theta | y \sim N\left(\frac{y\sigma^2 + m\sigma_y^2}{\sigma^2 + \sigma_y^2}, \sqrt{\frac{\sigma^2\sigma_y^2}{\sigma^2 + \sigma_y^2}}\right) \) is a normal distribution.

iii. Mean is \( \frac{y\sigma^2 + m\sigma_y^2}{\sigma^2 + \sigma_y^2} \), SD is \( \sqrt{\frac{\sigma^2\sigma_y^2}{\sigma^2 + \sigma_y^2}} \).
(c) i. Here, prior distribution of $\theta \sim N(m, \sigma)$. Hence,

$$p(\theta|y_1, \ldots, y_n) \propto p(\theta|y_1, \ldots, y_n)p(\theta)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \theta)^2}{2\sigma_y^2}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}$$

$$\propto \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \theta)^2}{2\sigma_y^2}\right\} \exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}$$

$$= \exp\left\{-\frac{\sum_{i=1}^{n}(y_i - \theta)^2}{2\sigma_y^2}\right\} \frac{\exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}}$$

$$= \exp\left\{-\frac{(n\sigma^2 + \sigma_y^2)\theta^2 - 2(\sum_{i=1}^{n} y_i\sigma^2 + m\sigma_y^2)\theta}{2\sigma_y^2}\right\}$$

$$\propto \exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\} \frac{\exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}}$$

If $Z \sim N(\mu, \sigma)$, then $p(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} \propto \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\}$. Here

$$p(\theta|y) \propto \exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\} \frac{1}{\sqrt{2\pi}\sigma_y^2} \exp\left\{-\frac{(\theta - m)^2}{2\sigma^2}\right\}$$

$$\frac{1}{\sqrt{2\pi}\sigma_y^2} \exp\left\{-\frac{(\theta - \sum_{i=1}^{n} y_i\sigma^2 + m\sigma_y^2)}{2\sigma_y^2}\right\}$$

ii. Refer to part (i), $\theta|y \sim N(\sum_{i=1}^{n} y_i\sigma^2 + m\sigma_y^2, \sqrt{\frac{\sigma_y^2\sigma_y^2}{n\sigma^2 + \sigma_y^2}})$ is a normal distribution.

iii. Mean is $\sum_{i=1}^{n} y_i\sigma^2 + m\sigma_y^2$, SD is $\sqrt{\frac{\sigma_y^2\sigma_y^2}{n\sigma^2 + \sigma_y^2}}$.

Ch2.Ex30 R codes;

```r
> pnorm(-0.72, -2, 1)
[1] 0.8997274
> pnorm(-0.36, -2, 1) - pnorm(-3.65, -2, 1)
[1] 0.900026
> 1 - pnorm(-3.28, -2, 1)
[1] 0.8997274
```

From the R output, we can see that the three intervals are 90% prediction intervals.
R codes;
> qnorm(0.8,-2,1)
[1] -1.158379
> qnorm(c(0.1,0.9),-2,1)
[1] -3.2815516 -0.7184484
> qnorm(0.2,-2,1)
[1] -2.841621

From the R output, we can see that the corresponding 80% prediction intervals are $(-\infty, -1.16]$, $[-3.28, -0.72]$, and $[-2.84, +\infty)$.

Ch.2-Ex35 Asiago: $n_A = 100$, $LS_{1.1}$ set
   Brie: $n_B = 400$, $LS_{1.1}$ set
   Cheshire: $n_C = 100$, $LS_{2}$ set

In general, notice that for any $\alpha \in (0, 1)$,

$$LS_\alpha = \{ \theta : l(\theta) \geq \alpha l(\hat{\theta}) \}$$

$$\approx \{ \theta : \exp\left\{ \frac{n(\theta - \hat{\theta})^2}{2\sigma^2} \right\} \geq \alpha \} \quad \text{(using CLT)}$$

$$= \{ \theta : |\theta - \hat{\theta}| \leq \sqrt{-2\log(\alpha)} \frac{\sigma}{\sqrt{n}} \}$$

Hence, $LS_\alpha \approx \left[ \hat{\theta} - \sqrt{-2\log(\alpha)} \frac{\sigma}{\sqrt{n}}, \hat{\theta} + \sqrt{-2\log(\alpha)} \frac{\sigma}{\sqrt{n}} \right]$ for any $\alpha \in (0, 1)$.

**Thus, length of $LS_\alpha$ is approximately** $\sqrt{-2\log(\alpha)} \frac{2\sigma}{\sqrt{n}}$

(a) For Asiago, $n_A = 100$ and $\alpha_A = 0.1$ and hence the length of $LS_{0.1}$ is approximately $\sqrt{-2\log(0.1)} \frac{2\sigma}{n_A} = 0.4292\sigma$.

For Brie, $n_B = 400$ and $\alpha_B = 0.1$ and hence the length of $LS_{0.1}$ is approximately $\sqrt{-2\log(0.1)} \frac{2\sigma}{n_B} = 0.2146\sigma$.

Hence, Asiago will have a longer interval and the interval will be about twice (b/c $0.4292/0.2146 = 2$) the length of that of Brie.

(b) For Cheshire, $n_C = 100$ and $\alpha_C = 0.2$ and hence the length of $LS_{0.2}$ is approximately $\sqrt{-2\log(0.2)} \frac{2\sigma}{n_C} = 0.3588\sigma$.

Hence, Asiago will have a longer interval and the interval will be about 1.2 times (b/c $0.4292/0.3588 = 1.196$) the length of that of Cheshire.

Ch.2-Ex36 $y$ is supposed to represent sample of size 15 from the exponential distribution with mean 100.

(lo, hi) is supposed to represent a 95% confidence interval for the population mean. estimated from the sample.

$n/1000$ is supposed to represent the approximate proportion of the times the interval estimate will contain the (true) population mean.

If a sample size of 15 is sufficiently large enough for the Central Limit Theorem to
apply, then approximately the value of $n/1000$ should be 0.95. Here is the result after running the loop:

```r
> n<-0
> for (i in 1:1000){
+ y<-rexp(15,0.01)
+ m<-mean(y)
+ s<-sqrt(var(y)/15)
+ lo<-m-2*s
+ hi<-m+2*s
+ if(lo<100 & hi>100 ) n<-n+1
+ }
> print(n/1000)
[1] 0.901
```

Remark: Note the correction that we made to the formula for $s<-\sqrt{\text{var}(y)/15}$, the division by 15 was missing. Also the end result of 0.901 is a random solution and may not match with your solution unless you fix the seed of random number generator.