1. Let $X$ have the log-normal distribution having the density
$$f(x) = \frac{1}{\sqrt{2\pi x}} e^{-(\log x)^2/2}, \quad 0 < x < \infty.$$ 
Find the density function of $Y = \sqrt{X}$.

2. Let $X$ have the standard Cauchy density function
$$f(x) = \frac{1}{\pi (1 + x^2)}, \quad -\infty < x < \infty.$$ 
Find the density function of $Y = 1/X$ and the median of $Z = Y^2 = 1/X^2$.

3. Let $(X,Y)$ have joint density $f(x,y) = x^{-2}y^{-2}I\{x > 1, y > 1\}$. Compute the joint density of $U = XY$ and $V = X/Y$. Also obtain the marginal density of $U$. [You may find it useful that $U^{-1} < V < U$.]

4. Let the joint density of $(X,Y)$ be
$$f(x,y) = \frac{1}{8}(x^2 - y^2)e^{-x}, \quad 0 < x < \infty, \quad -x < y < x.$$ 
Find the covariance of $X$ and $Y$.

5. Let $X_1, X_2, X_3, X_4$ be independent $N(0,1)$. Show that
$$T = \frac{\sqrt{3}(X_1 + X_2 + X_3 + X_4)}{\sqrt{(X_1 - X_2 + X_3 - X_4)^2 + (X_1 + X_2 - X_3 - X_4)^2 + (X_1 - X_2 - X_3 + X_4)^2}}$$
has $t$-distribution with 3 degrees of freedom. Calculate the variance of $T$.

6. Let $X_1, X_2, X_3, X_4$ be i.i.d. $N(0,1)$. Show that
$$\frac{(X_1 + X_2 + X_3 + X_4)^2 + (X_1 - X_2 + X_3 - X_4)^2}{(X_1 + X_2 - X_3 - X_4)^2 + (X_1 - X_2 - X_3 + X_4)^2}$$
has $F$ distribution with $(2,2)$ degrees of freedom.

7. Let $X_1, \cdots, X_n$ be $n$ independent observations from the density $f(x) = \frac{1}{2}x^2e^{-x}$ and let $T_n = \frac{\sum_{i=1}^{n}X_i^2}{n}$. Find constants $a$ and $b$ such that $\sqrt{n}(T_n - a)/b$ converges in distribution to $N(0,1)$ as $n \to \infty$. [Hint: Consider $Y_i = X_i^{-1}$ and use the delta method.]
8. Let \(X_1, \ldots, X_n\) be \(\text{Normal}(0, \theta)\) and \(Y_1, \ldots, Y_n\) be \(\text{Normal}(0, 1)\), and let all the variables be mutually independent. Consider \(V_n = \frac{X_1^2 + \cdots + X_n^2}{Y_1^2 + \cdots + Y_n^2}\).

(a) Show that 
\[E(V_n) = \frac{n\theta}{n-2}.\]

In this derivation, the fact that 
\[\int_0^\infty x^{a-1}e^{-x/b}dx = b^a\Gamma(a), \quad \text{for all } a > 0, b > 0,\]
may be useful.

(b) Using the fact that 
\[V_n - \theta = \frac{\sum_{i=1}^n (X_i^2 - \theta Y_i^2)}{\sum_{i=1}^n Y_i^2},\]
show that \(\sqrt{n}(V_n - \theta)\) converges in distribution to \(\text{Normal}(0, 4\theta^2)\).

(c) Is it true that \(\sqrt{n}(V_n - E(V_n))\) converges in distribution to \(\text{Normal}(0, 4\theta^2)\)? Justify your answer.

(d) Obtain the asymptotic distribution of \(\log V_n\).

9. A shooter hits a target with probability \(p\) independently in each attempt. She decides to hit the target \(r\) times. Let \(X\) stand for the (random) number of attempts she needs. From the first principle, derive the probability mass function of \(X\).

Find the moment generating function \(m(t) = E(e^{tX})\) of \(X\). For what values of \(t\) will \(m(t)\) be finite?

If \(r \to \infty\) and \(p \to 1\) such that \(r(1-p) \to \lambda\), where \(0 < \lambda < \infty\), show that the distribution of \(X - r\) can be approximated by the Poisson distribution with parameter \(\lambda\). [Hint: Calculate the moment generating function of \(X - r\) and identify its limit.]

If \(r \to \infty\) and \(0 < p < 1\) remains fixed, show that the distribution of \(r^{-1/2}(X - r/p)\) can be approximated by a normal distribution. What are the mean and variance of the approximating normal distribution? Clearly mention any theorem you are using.