Notations: Let \((X_i, \Delta_i), i = 1, \ldots, n\), be a sample of i.i.d. censored survival data, where \(X_i = T_i \wedge C_i\) and \(\Delta_i = I(T_i \leq C_i)\). We assume independence censoring assumption, i.e. \(T_i \perp C_i\). In addition, we assume that all the failure times and censoring times are continuous. Define the counting process \(N_i(t) = I(X_i \leq t, \Delta_i = 1)\) and the at-risk process \(Y_i(t) = I(X_i \geq t)\).

1. Define the filtration

\[ F^D(t) = \sigma\{I(X_i \leq u, \Delta_i = 1) : u \leq t, i = 1, \ldots, n\}. \]

That is, \(F^D(t)\) is all the information up to and including time \(t\) of the observed deaths in a sample of size \(n\).

a. Derive the intensity process \(a^D(t)\) of \(N(t)\) with respect to the filtration \(\{F^D(t)\}\).

b. Consider the \(\{F^D(t)\}\)-martingale process \(M^D(t) = N(t) - \int_0^t a^D(u)du\). Consider the stochastic process \(Z(t) = \int_0^t Y(u)dM^D(u)\). Is \(Z(t)\) an \(\{F^D(t)\}\)-martingale process? Why or why not?

c. Define by \(N(t^-)\) the left-continuous version of the counting process \(N(t)\). Consider the stochastic processes \(Z^*(t) = \int_0^t N(u^-)dM^D(u)\). Is \(Z^*(t)\) an \(\{F^D(t)\}\)-martingale process? Why or why not?

d. Derive the predictable quadratic variation process of \(Z^*(t)\), i.e. \(<Z^*(t), Z^*(t)>\).

e. Find an unbiased estimator of \(\text{var}\{Z^*(t)\}\).

2. Let \((X_i, \Delta_i, Z_i), i = 1, \ldots, n\), be a sample of i.i.d. censored survival data, where \(Z_i\) is a \(p\)-dimensional vector of covariates. Here we assume the conditional independence censoring assumption, i.e. the failure time \(T\) is independent of the censoring time \(C\) given covariates \(Z\). Let \(\lambda(t|z)\) denote the hazard function of \(T\) for a subject with covariate \(Z = z\), i.e.

\[ \lambda(t|z) = \lim_{h \downarrow 0} P(t \leq T < t + h|T \geq t, Z = z)/h. \]
Define the filtration

\[ F(t) = \sigma\{I(X_i \leq u, \Delta_i = 1), I(X_i \leq u, \Delta_i = 0), Z_i : u \leq t, i = 1, \ldots, n\}. \]

That is, \( F(t) \) is all the information up to and including time \( t \) of the observed deaths, censoring and covariates in a sample of size \( n \). Show that the intensity process of the counting process \( N_i(t) \) with respect to \( F(t) \) is given by \( a_i(t) = Y_i(t)\lambda(t|Z_i) \). (Thus, \( M_i(t) = N_i(t) - \int_0^t Y_i(u)\lambda(u|Z_i)du \) is a \( \{F(t)\} \)-martingale process, \( i = 1, \ldots, n \)).

3. To characterize the individual heterogeneity, consider an unobserved random effect, \( \alpha_i \), also commonly referred to as a frailty variate in multivariate survival analysis. Let \( \alpha_i \lambda_i(t) \) denote the conditional hazard function of \( T_i \) given \( \alpha_i \), i.e. \( P(T_i > t|\alpha_i) = \exp\{-\alpha_i\Lambda(t)\} \), where \( \Lambda(t) = \int_0^t \lambda(s)ds \).

Define two filtrations

\[ F_i(t) = \sigma\{I(X_i \leq u, \Delta_i = 1), I(X_i \leq u, \Delta_i = 0), \alpha_i : u \leq t\}, \]
\[ F^*_i(t) = \sigma\{I(X_i \leq u, \Delta_i = 1), I(X_i \leq u, \Delta_i = 0) : u \leq t\}. \]

a. Derive the conditional intensity process \( a_i(t) \) of \( N_i(t) \) with respect to the filtration \( \{F_i(t)\} \).

b. Let \( a^*_i(t) \) denote the intensity process of \( N_i(t) \) with respect to the filtration \( \{F^*_i(t)\} \). Derive an expression for \( a^*_i(t) \) as a function of \( a_i(t) \).