Applied Multivariate and Longitudinal Data Analysis

Review of vectors and matrices & R Implementation

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1 Vectors

Let us start by defining a vector $x \in \mathbb{R}^p$. In this course, we will always take a vector as a column vector by convention (unless otherwise specified). Specifically, we will write

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix}.$$ 

A row vector is written as

$$x^T = (x_1, x_2, \ldots, x_p).$$

Note that the notation $x^T$ denotes “transpose” of $x$.

Example: Consider the following data table (available in R) providing quarterly earnings (first two quarters) in US dollars per Johnson & Johnson share during 1960 - 1964 (available in R).

```r
> JohnsonJohnson
     Qtr1 Qtr2
1960 0.71 0.63
1961 0.61 0.69
1962 0.72 0.77
1963 0.83 0.80
1964 0.92 1.00
```

Now consider the earnings of quarter 1 over the years 1960 - 1964. This is an example of a $5 \times 1$ (column) vector

$$x = \begin{pmatrix} 0.71 \\ 0.61 \\ 0.72 \\ 0.83 \\ 0.92 \end{pmatrix}.$$ 

To create this vector in R, we can use the command:

```r
> x = c(0.71, 0.61, 0.72, 0.83, 0.92)
```

Try printing $x$:

```r
> x
[1] 0.71 0.61 0.72 0.83 0.92
```
R prints it using a single line but it still considers x as a column vector. To see this, try to view x in a matrix form using `as.matrix()`:

```R
as.matrix(x)
[,1]
[1,] 0.71
[2,] 0.61
[3,] 0.72
[4,] 0.83
[5,] 0.92
```

Try to take the transpose using `t()`:

```R
> t(x)
[1,] 0.71 0.61 0.72 0.83 0.92
```

### 1.1 Addition and subtraction of two vectors

For two vectors \( a, b \in \mathbb{R}^p \), the sum is defined as:

\[
\begin{pmatrix}
a_1 + b_1 \\
a_2 + b_2 \\
\vdots \\
a_p + b_p
\end{pmatrix},
\]

that is, a vector of same length as of \( a \) and \( b \), where each element is the sum of corresponding elements of \( a \) and \( b \). We can easily define \( a - b \) in a similar fashion.

**Example:** Consider the earnings of quarters 1 and 2 over the years 1960 - 1964 in the Johnson & Johnson data set.

```R
> a = c(0.71, 0.61, 0.72, 0.83, 0.92)
> b = c(0.63, 0.69, 0.77, 0.80, 1.00)

Total earnings over the first 2 quarters is

```R
> a+b
[1] 1.34 1.30 1.49 1.63 1.92
```

Difference between earnings between first 2 quarters is

```R
> a-b
[1] 0.08 -0.08 -0.05 0.03 -0.08
```
1.2 Vector multiplication

Here we will learn about the so called *inner product* of two vectors. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$, the inner product is defined as:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T \mathbf{b} = \sum_{j=1}^{p} a_j b_j = \mathbf{a}^T \mathbf{b}.$$ 

Note that the result is a scalar. In a more detailed view

$$\mathbf{a}^T \mathbf{b} = \begin{pmatrix} a_1 & a_2 & \ldots & a_p \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} = a_1 b_1 + a_2 b_2 + \ldots + a_p b_p$$

Note that to take the inner product, the two vectors must have the same number of elements.

**Example:** Suppose $\mathbf{a}^T = (1, 0, 2, 5)$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 6 \end{pmatrix}$ we have:

$$\begin{pmatrix} 1 & 0 & 2 & 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 1 \\ 6 \end{pmatrix} = 1 \times 2 + 0 \times 3 + 2 \times 1 + 5 \times 6 = 34$$

In R the inner product (or in general matrix multiplication) can be performed by using `%*%`. In this example,

```r
> a <- c(1, 0, 2, 5)
> b <- c(2, 3, 1, 6)
> t(a) %% b
[1,1] 34
```

Be careful to use `%*%`. Be sure to put the `%` signs properly. Just using `*` without the `%` signs would give you wrong result.

```r
> a <- c(1, 0, 2, 5)
> b <- c(2, 3, 1, 6)
> t(a)*b
```


What are we getting if we do not put the % signs properly?

1.3 Vector length

The length of a vector is defined as its distance from the vector 0, the origin. It is defined as

\[ |x| = \langle x, x \rangle^{1/2} = (x_1^2 + \ldots + x_p^2)^{1/2}. \]

In other words, the length of a vector \( x \) is the square root of the inner product of \( x \) with itself. Try to compute length of \( x \) defined before:

```r
> sqrt(sum(x^2))
[1] 4.821825
```

1.4 Orthogonal vectors

Two vectors \( a \) and \( b \) (that have the same number of elements) are said to be orthogonal if \( a^T b = 0 \). In other words, two vectors are orthogonal if their inner product is zero.

Two vectors \( a \) and \( a \) are orthonormal if their inner product is zero and each of them has length 1, that is, if \( a^T b = 0 \), and \( a^T a = 1 \) and \( b^T b = 1 \) then they are orthonormal.

Example: Recall the vectors \( a \) and \( b \) defined before. Are they orthogonal? Are they orthonormal?
2 Matrices

In this section, we learn about matrices. Simply put, matrices are array (table) of numbers. For example, consider Johnson & Johnson data set as before:

\begin{verbatim}
> JohnsonJohnson
   Qtr1  Qtr2
1960 0.71 0.63
1961 0.61 0.69
1962 0.72 0.77
1963 0.83 0.80
1964 0.92 1.00
\end{verbatim}

This is an example of a $5 \times 2$ matrix, where each column corresponds to a quarter and each row corresponds to a particular year. We can write this matrix as

\begin{equation*}
M = \begin{pmatrix}
0.71 & 0.63 \\
0.61 & 0.69 \\
0.72 & 0.77 \\
0.83 & 0.80 \\
0.92 & 1.00
\end{pmatrix};
\end{equation*}

the size of the matrix $M$ is $5 \times 2$ as it has 5 rows and 2 columns. In general, a matrix can have any number of rows and columns.

To create the matrix $M$ in R, and then to print, we use the following commands:

\begin{verbatim}
> M = cbind(c(0.71, 0.61, 0.72, 0.83, 0.92), c(0.63, 0.69, 0.77, 0.80, 1.00))
> M
[,1] [,2]
[1,] 0.71 0.63
[2,] 0.61 0.69
[3,] 0.72 0.77
[4,] 0.83 0.80
[5,] 0.92 1.00
\end{verbatim}

This way of creating matrix is essentially taking each column and then joining them together.

One could also try:

\begin{verbatim}
> mydata = c(0.71, 0.61, 0.72, 0.83, 0.92, 0.63, 0.69, 0.77, 0.80, 1.00)
> M = matrix(mydata, nrow=5, ncol=2, byrow=F)
\end{verbatim}
By default, the command `matrix()` fills the matrix by columns. One could try to fill the matrix by rows by including the argument `byrow = TRUE` in the call to `matrix`.

One could also read the matrix into R from an external file:

```r
> M = read.table(file="mydata.txt", header=FALSE)
```

where `mydata.txt` is an external file containing the values of the matrix with no column names (and hence `header=FALSE`). If column names are included in the file on top of each column, then use `header=TRUE` in the argument.

### 2.1 Some special matrices

There are some matrices which have particular structure or properties of interest. We will use the following matrices often in this course.

#### 2.1.1 Identity Matrix

An identity matrix (of any size), is a diagonal matrix with 1 as each diagonal entry. For example, $I_3$ is defined as

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
2.1.2 Ones
We also need to define a vector of ones; \(1_p\), a \(p \times 1\) matrix containing only the value 1. There is no inbuilt function in R to create this vector, it is easily added:

```R
> ones <- function(p){
  temp <- matrix(1,p,1)
  return(temp)
}
> ones(3)
[,1]
 [1,]    1
 [2,]    1
 [3,]    1
```

2.1.3 Zero matrix
Finally, \(0\) denotes the zero matrix, a matrix of zeros. Unlike the previously mentioned matrices this matrix can be any shape you want. So, for example:

\[
0_{2 \times 3} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

```R
> matrix(0, nrow=2,ncol=3)
 [,1] [,2] [,3]
[1,]   0   0   0
[2,]   0   0   0
```

2.1.4 Diagonal Matrices
A diagonal matrix is a square matrix in which all the “off diagonal” elements are zero. An example of diagonal matrix is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}
\]

```R
> diag(c(1:3))
 [,1] [,2] [,3]
[1,]  1  0  0
[2,]  0  2  0
[3,]  0  0  3
```
2.1.5 Symmetric matrices

A matrix \( A \) is called a symmetric matrix if \( A_{ij} = A_{ji} \), that is, \( A = A^T \). As a consequence, symmetric matrix has to square, that is, they has to have the same number of rows and columns. For example, the following is a symmetric matrix:

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{pmatrix}.
\]

2.1.6 Idempotent matrices

It may be noted that \( H \) is idempotent, i.e. \( H = H^T \) and \( H = H^2 \).

2.2 Addition and subtraction

Addition and subtraction is possible only if the matrices have the same size. Just like vectors, the sum of two matrices \( A \) and \( B \) (of same size) is another matrix (of the same size) where each element is the sum of the corresponding elements of \( A \) and \( B \).

\[
A = \begin{pmatrix}
0.71 & 0.61 & 0.72 & 0.83 & 0.92 \\
0.63 & 0.69 & 0.77 & 0.80 & 1.00
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
6 & 7 & 8 & 9 & 10
\end{pmatrix}
\]

\[
A + B
\]
A little more care is needed in defining basic mathematical operations on matrices. Considering the two matrices $A$ and $B$, we consider their equality $A = B$ if any only if:

- $A$ and $B$ have the same size, and
- the $(i,j)$-th element of $A$ is equal to the $ij$th element of $A$ for all $1 \leq i \leq r$ and $1 \leq j \leq n$

The following two zero matrices are not equal:

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\neq
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
$$

Matrix addition follows all the normal arithmetic rules, i.e.

- **Commutative law** $A + B = B + A$
- **Associative law** $A + (B + C) = (A + B) + C$

### 2.3 Multiplication

Multiplication of a matrix by a scalar: multiply every element in the matrix by the scalar.

So if $k = 0.4$, and

$$
A = \begin{pmatrix}
1 & 5 & 8 \\
1 & 2 & 3 \\
\end{pmatrix}
$$

we can calculate $kA$ as:

$$
kA = 0.4 \times \begin{pmatrix}
1 & 5 & 8 \\
1 & 2 & 3 \\
\end{pmatrix} = \begin{pmatrix}
0.4 & 2 & 3.2 \\
0.4 & 0.8 & 1.6 \\
\end{pmatrix}
$$

Matrix multiplication however follows vector multiplication, and therefore does not follow the same rules as basic multiplication.
To multiply two matrices $A$ and $B$, one must first check that the number of columns in $A$ is exactly the same as the number of rows in $B$. Otherwise, we cannot multiply these two matrices. More generally, multiplication proceeds with the matrix size as follows:

$$A_{m \times n} \times B_{n \times p} = C_{m \times p}.$$

Let $A$ be of size $m \times n$; represent $A$ using its row vectors $a_1^T, a_2^T, \ldots, a_m^T$. Let $B$ be of size $n \times p$; represent $B$ using its column vectors $b_1, b_2, \ldots, b_p$. The multiplication operation for matrices is defined as:

$$AB = \begin{pmatrix}
a_1^T \\
a_2^T \\
\vdots \\
a_m^T
\end{pmatrix} \begin{pmatrix}
b_1 & b_2 & \cdots & b_p
\end{pmatrix} = \begin{pmatrix}
a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\
a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\
\vdots & \vdots & \ddots & \vdots \\
a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p
\end{pmatrix}$$

In other words, we need to multiply row $i$ of $A$ by column $j$ of $B$ to give element $ij$ of the result.

Consider the following example.

```r
> A = cbind(c(0.71, 0.61, 0.72, 0.83, 0.92), c(0.63, 0.69, 0.77, 0.80, 1.00))
> A
 [,1] [,2]
[1,] 0.71 0.63
[2,] 0.61 0.69
[3,] 0.72 0.77
[4,] 0.83 0.80
[5,] 0.92 1.00
> B = matrix(c(1,2,3,4,5,6,7,8,9,10),2,5)
> B
[1,] 1  3  5  7  9
[2,] 2  4  6  8 10
```

Here $A$ has 2 columns and $B$ has 2 rows, and hence we can multiply $A$ with $B$. We write this as $C = AB$. The resulting matrix $C$ is defined such that the $(i, j)$-th element of $C$ is the inner product between the $i$-th row of $A$ and the $j$-th row of $B$. In R, we only need to use the `%*%` operator to ensure we are getting matrix multiplication:

```r
> C = A %*% B
```
Just to check, look at $C_{23}$, the (2,3)-th element of $C$.

$$C_{23} = 7.19 = (0.61, 0.69) \begin{pmatrix} 5 \\ 6 \end{pmatrix} = 5 \times 0.61 + 6 \times 0.69.$$  

It is particularly important to use the correct *matrix multiplication* argument. Depending on the matrices you are working with (if they both have the same dimensions), using the usual $*$ multiplication operator will give you the Hadamard product, the element by element product of the two matrices which is rarely what you want:

```r
> B <- matrix(c(1,3,4,2),2,2)
> B
 [,1] [,2]
[1,] 1 4
[2,] 3 2

> C <- matrix(c(1,1,3,5),2,2)
> C %*% B ## correct call for matrix multiplication
 [,1] [,2]
[1,] 10 10
[2,] 16 14

> C * B ## Hadamard Product!!!
 [,1] [,2]
[1,] 1 12
[2,] 3 10
```

Note that you can’t multiply non-conformable matrices; this is one place in R where you get a clearly informative error message:

```r
> B %*% A
Error in B %*% A : non-conformable arguments
```

We saw earlier that matrix addition was commutative and associative. But as you can
imagine, given the need for comfortability some differences may be anticipated between conventional multiplication and matrix multiplication. Generally speaking, matrix multiplication is not commutative (you may like to think of exceptions):

\[(\text{non-commutative}) \quad A \times B \neq B \times A\]

**Associative law** \[A \times (B \times C) = (A \times B) \times C\]

And the distributive laws of multiplication over addition apply as much to matrix as conventional multiplication:

\[A \times (B + C) = (A \times B) + (A \times C)\]

\[(A + B) \times C = (A \times C) + (B \times C)\]

But there are a few pitfalls if we start working with transposes. Whilst

\[(A + B)^T = A^T + B^T\]

note that:

\[(A \times B)^T = B^T \times A^T\]

### 2.4 Other operations with matrices

#### 2.4.1 Trace of a matrix

The trace is only defined for a square matrix, that is, a matrix having same number of rows and columns. The trace of a matrix is the sum of its diagonal elements. There is no inbuilt function in R to calculate this you need to create your own function.

```r
trace <- function ( A ){
  temp = (nrow(A)== nrow(A))
  if(temp) {temp = sum(diag(A))}
  return(temp) }
```

```r
> B <- matrix(c(1,3,4,2),2,2)
> B
```
Note that if you have two conformable matrices $A$ e.g. \[
\begin{pmatrix}
2 & 5 \\
0 & 7 \\
4 & 3
\end{pmatrix}
\] and $B$ e.g. \[
\begin{pmatrix}
4 & 2 & 1 \\
6 & 3 & 2
\end{pmatrix}
\],
\[\text{trace}(AB) = \text{trace}(BA)\]

### 2.4.2 Transpose

Transposing matrices simply involves turning the first column into the first row, second column into second row and so on. A transposed matrix is denoted by a superscripted $T$, in other words $M^T$ is the transpose of $M$.

If \[M = \begin{pmatrix}
1 & 4 \\
2 & 5 \\
3 & 6
\end{pmatrix}\] then \[M^T = \begin{pmatrix}
1 & 2 & 3 \\
2 & 5 & 6
\end{pmatrix}\]

We can use `t()` to take a transpose in $\mathbb{R}$:

```r
> Mt = t(M)
> Mt
[,1] [,2] [,3]
[1,]  1  2  3
[2,]  4  5  6
> dim(Mt)
[1] 2 3
```

The dimensions of any matrix can be checked with `dim()`.

```r
> dim(M)
[1] 3 2
> dim(Mt)
[1] 2 3
```

We can also access certain elements of the matrix. For example $M_{12}$ denotes the element of $M$ with is in the 1st row and 2nd column of the matrix; in this case, $M_{12} = 4$. 

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2.4.3 Powers of matrices

Let \( A \) be a square matrix (number of its rows equals the number of columns). We defined \( A^0 = I \), the identity matrix and \( A^1 = A \). Then define \( A^2 = AA \). Using these definitions for matrix powers means that all the normal power arithmetic applies. For example, \( A^m \times A^n = A^{m+n} \). If you look closely, you can also see that the powers of a matrix are commutative which means that we can do fairly standard algebraic factorisation. For example:

\[
\]

2.5 Determinants

The determinant of a \( p \times p \) matrix \( A \) is denoted as \( |A| \). Finding the determinant of a \( 2 \times 2 \) matrix is easy:

\[
|A| = \det \begin{pmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

For matrices of order \( \geq 2 \) the determinant of \( A \) is calculated through a somewhat more complex formula.

In \( \mathbb{R} \), \texttt{det()} tries to find the determinant of a matrix.

```r
> D <- matrix(c(5,3,9,6),2,2)
> D
[,1] [,2]
[1,]  5  9
[2,]  3  6
> det(D)
[1] 3

> E <- matrix(c(1,2,3,6,6,7,8,9,10),3,3)
> E
[,1] [,2] [,3]
[1,]  1  6  8
```
Some useful properties of determinants:

- The determinant of a diagonal matrix (or a triangular matrix for that matter) is the product of the diagonal elements.
- For any scalar $k$, $|kA| = k^n|A|$, where $A$ has size $n \times n$.
- If two rows (or columns) of a matrix are interchanged, the sign of the determinant changes.
- If two rows or columns are equal or proportional (see material on rank later), the determinant is zero.
- The determinant is unchanged by adding a multiple of some column (row) to any other column (row).
- If all the elements or a column / row are zero then the determinant is zero.
- If two $n \times n$ matrices are denoted by $A$ and $B$, then $|AB| = |A|.|B|$.

2.6 Rank of a matrix

Rank denotes the number of linearly independent rows or columns. For example:

$$
\begin{pmatrix}
1 & 1 & 1 \\
2 & 5 & -1 \\
0 & 1 & -1
\end{pmatrix}
$$

is $3 \times 3$ dimensional matrix with rank 2. The second column $a_2$ can be found from the other two columns as $a_2 = 2a_1 - a_3$.

If all the rows and columns of a square matrix $A$ are linearly independent it is said to be of full rank and non-singular. If $A$ is singular, then $|A| = 0$. 
3 Matrix inversion

If $A$ is a non-singular $p \times p$ matrix, then there is a unique matrix $B$ such that $AB = BA = I$, where $I$ is the identity matrix given earlier. In this case, $B$ is the inverse of $A$, and denoted $A^{-1}$.

If $A$ is $2 \times 2$ matrix then its inverse can be calculated as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ then } A^{-1} = \frac{1}{|A|} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

More generally for a matrix of order $n \times n$, the $(j,k)$-th entry of $A^{-1}$ is given by:

$$\left[ \frac{|A_{-jk}|}{|A|} \right]^{(-1)^{j+k}},$$

where $A_{-jk}$ is the matrix formed by deleting the $j$th row and $k$th column of $A$. Note that a singular matrix has no inverse since its determinant is 0.

In R, we use `solve()` to invert a matrix (or solve a system of equations if you have a second matrix in the function call, if we don’t specify a second matrix R assumes we want to solve against the identity matrix, which mean finding the inverse).

```r
> D <- matrix(c(5,3,9,6),2,2)
> solve(D)
     [,1]       [,2]
[1,]   2 -3.000000
[2,] -1  1.666667
```

Some properties of inverses:

- The inverse of a symmetric matrix is also symmetric.
- The inverse of the transpose of $A$ is the transpose of $A^{-1}$.
- The inverse of the product of several square matrices is a little more subtle: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. If $c$ is a non-zero scalar then $(cA)^{-1} = c^{-1}A^{-1}$.
- The inverse of a diagonal matrix is really easy - the reciprocals of the original elements.
4 Eigenvalues and eigenvectors

If $A$ is a square $p \times p$ matrix, the eigenvalues (latent roots, characteristic roots) are the roots of the equation:

$$|A - \lambda I| = 0;$$

this (characteristic) equation is a polynomial of degree $p$ in $\lambda$. The roots, the eigenvalues of $A$, are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$. For each eigenvalue $\lambda_i$ there is a corresponding eigenvector $e_i$ which can be found by solving:

$$(A - \lambda_i I)e_i = 0$$

There are many solutions for $e_i$. For our (statistical) purposes, we usually set it to have unit length ($|e_i| = 1$). The normalized vector corresponding to some vector $a$ is defined by $a = v / \sqrt{v^T v}$

Some important results:

(a) $\text{trace}(A) = \sum_{i=1}^{p} \lambda_i$

(b) $|A| = \prod_{i=1}^{p} \lambda_i$

Also, if $A$ is symmetric:

(c) The normalized eigenvectors corresponding to unequal eigenvalues are orthonormal

(d) If the matrix $A$ is positive definite (that is $x^T A x > 0$ for all $x \neq 0$) then all the eigenvalues are positive;

(e) If the matrix $A$ is positive semi-definite (that is $x^T A x \geq 0$ for all $x \neq 0$ and the equality is obtained for non-zero vectors $x$) then all the eigenvalues are non-negative. The number of non-zero eigenvalues is equal to the rank of the matrix.

(f) Correlation and covariance matrices: are symmetric positive definite (or semi-definite).

If such a matrix is of full rank $p$ then all the eigenvalues are positive. If the matrix is of rank $m < p$ then there will be $m$ positive eigenvalues and $p - m$ zero eigenvalues.

In R these decompositions are carried using the `eigen()` function.
Let $A$ be $p \times p$ square matrix which is positive semi-definite. Denote by $\Lambda$ the $p \times p$ diagonal matrix of the eigenvalues $\lambda_i$’s, $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_p)$, and let $E = (e_1|e_2|\ldots,e_p)$ the $p \times p$ matrix of the corresponding eigenvectors $e_i$.

Then we have the representation (also known by spectral decomposition):

$$A = E \Lambda E^T.$$  \hfill (1)

The square root of $A$ is defined by $A^{1/2} = E \Lambda^{1/2} E^T$, where $\Lambda^{1/2} = diag(\lambda_1^{1/2}, \lambda_2^{1/2}, \ldots, \lambda_p^{1/2})$.

Properties: $A^{1/2} A^{1/2} = A$