Chapter 7.3: Intervals Based on a Normal Population Distribution

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Outline of this chapter:

• Properties of $t$ distribution
• Confidence Intervals using $t$ distribution
• Prediction interval
Review of Chapters 7.1 and 7.2

1. A confidence interval for population mean $\mu$ when
   - population distribution is normal
   - population standard deviation $\sigma$ is known

2. Large-sample confidence interval for population mean $\mu$ and proportion $p$ when
   - population distribution is unknown
   - population standard deviation $\sigma$ is unknown

Large sample intervals (Chapter 7.2) are useful when the population distribution is unknown. However, when the sample size $n$ is small, we cannot use such large sample intervals. In this situation, we need to make a specific assumption about the form of the population distribution and then derive a CI tailored to that assumption.

**Assumption:** The population distribution is Normal and both the population mean $\mu$ and population variance $\sigma^2$ are unknown.

- Suppose $X_1, \ldots, X_n$ are random samples from a $N(\mu, \sigma^2)$ distribution, where both $\mu$ and $\sigma$ are unknown.

- **Result:** The quantity
  \[
  T = \frac{\bar{X} - \mu}{S/\sqrt{n}}
  \]
  has a $t$ distribution with $n - 1$ degrees of freedom.
Properties of $t$ distribution

- We denote a $t$ distribution with $\nu$ degrees of freedom as $t_\nu$.
- The density of any $t$ distribution is a bell shaped curve centered at zero.
- Each $t$ density curve is more spread out than the standard normal, that is the $N(0,1)$ density. This is especially true when the degrees of freedom $d$ is small.
- As degrees of freedom becomes larger and larger, the spread of $t$ density curves becomes smaller.
- If the degrees of freedom $\nu$ is “large enough”, the $t_\nu$ density curve becomes very close the $N(0,1)$ density curve.

To compute confidence intervals, we need the critical values of $t$ distributions:

Typically, as $t$ density curves have more spread than the standard normal curves, the critical values from $t$ distributions are different than those from $N(0,1)$ distribution.

**Example:** Comparison between $t$ critical values versus $N(0,1)$ critical values

- For a 95% two-sided CI, we have $\alpha = 0.05$ so that $\alpha/2 = 0.025$. Suppose degrees of freedom $\nu = 5$. Then we have
  \[ t_{0.025, 5} = 2.55 \quad \text{and} \quad z_{0.025} = 1.96 \]

- For a 99% two-sided CI, we have $\alpha = 0.01$ so that $\alpha/2 = 0.005$. Suppose degrees of freedom $\nu = 5$. Then we have
  \[ t_{0.005, 5} = \quad \text{and} \quad z_{0.005} = \]
A *t* confidence interval for $\mu$

- Suppose $X_1, \ldots, X_n$ are random samples from a $N(\mu, \sigma^2)$ distribution, where both $\mu$ and $\sigma$ are unknown.
  - Estimate $\mu$ by sample mean $\bar{X}$
  - Estimate $\sigma$ by sample standard deviation $S$

- A $100(1 - \alpha)$% *t*-confidence interval for $\mu$ can be formed as

$$
(\bar{X} - t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}, \bar{X} + t_{\alpha/2, n-1} \frac{S}{\sqrt{n}})
$$

  - Note that we use the *t* critical values with $n - 1$ degrees of freedom in the formula above.

- This formula is valid regardless of the sample size $n$. The crucial assumption is that the population distribution is normal.

- Interval width: $2t_{\alpha/2, n-1} \frac{S}{\sqrt{n}}$

**Question:** Consider the CI based on standard normal critical values versus the *t* interval described above. When sample size is small, which interval do you expect to be wider?
1. A research engineer for a tire manufacturer is investigating the tire life for a new rubber compound. She has built 10 tires and tested them to end-of-life in a road test. From the sample, the mean was 61,692 kilometers and the standard deviation was 3035 kilometers. Assume that the life time of the tire is normally distributed.

- Find a 90% confidence interval for the average kilometers to end-of-life for all of the new tires. State the assumptions that are met and interpret the calculated interval in the context of the problem.

- For the sake of demonstration, find a 90% large-sample confidence interval for the average kilometers to end-of-life for all of the new tires.

- Compare the two intervals.
2. Allowable mechanical properties for structural design of metallic aerospace vehicles requires an approved method for statistically analyzing empirical test data. The article Establishing Mechanical Property Allowables for Metals (J. of Testing and Evaluation, 1998: 293-299) used the data provided in Exercises 1.13 on tensile ultimate strength (ksi) as a basis for addressing the difficulties in developing such a method.

We have the following descriptive statistics output.

\[ n = 153, \quad \bar{x} = 135.39, \quad s = 4.59, \quad \text{Minimum} = 122.20, \quad \text{Maximum} = 147.70. \]

- Assume that the population distribution is normal. Calculate a 95% confidence interval for the true average ultimate tensile strength.

- Compute the large sample 95% confidence interval for the true average ultimate tensile strength.

- Compare the two intervals computed above.
3. Choose the appropriate CI for each of the following questions and state the assumptions that are met. Explain your answer.

- The alternating current (AC) breakdown voltage (the minimum voltage that makes an insulator react as a conductor) of an insulating liquid indicates its dielectric strength. A sample measuring the breakdown voltage (kV) of a particular circuit gives \( n=48, \sum_{i=1}^{48} x_i = 2626, \) and \( \sum_{i=1}^{48} x_i^2 = 144,950. \) Construct a 95% interval for the true mean breakdown voltage is of interest.

- A random sample of 40 lightning flashes in a certain region resulted in an average radar echo duration of 0.81 sec. The standard deviation is estimated to be 0.34 sec. Construct a 90% CI for the true average echo duration is of interest.

- Data is collected on 25 automobiles and the average gas mileage of the sample is 23.5 mpg with a standard deviation of 7.82 mpg. Assume the gas mileage is normally distributed. Using a 90% confidence level, construct an interval for the mean gas mileage of all automobiles is of interest.

- A biology class is asked to find the average wingspan of monarch butterflies. The class caught and measured the wingspans of 46 monarch butterflies. The average length (in millimeters) was found to be 93.5 with a standard deviation of 3.44mm. Obtain a 99% confidence interval for the wingspan of all monarch butterflies is of interest.

A good exercise would be to make sure you can find all of these intervals and interpret them in the context of the problem.
A Prediction Interval for a Single Future Value

In many applications, the objective is to predict a single value of a variable to be observed at some future time, rather than to estimate the mean value of that variable. This situation is different from estimating a parameter (e.g., population mean) because the future observation is in fact a random variable (not a parameter).

**Example:** Book example 7.12

- Sample mean \( \bar{X} \) can be used as a point prediction of a future value
- A 95% confidence interval only provides you information about the overall mean only. Specifically, it tells you how precise the estimate of population mean is.
- This does not provide any information about how reliable the predicted value of a future observation is.

**Setup:** We have a random sample \( X_1, X_2, \ldots, X_n \) from a normal population distribution with unknown mean \( \mu \) and unknown variance \( \sigma^2 \). We wish to predict the value of \( X_{\text{new}} \), a single future observation coming from the same distribution.

A point predictor is \( \bar{X} \). Thus the prediction error is \( \bar{X} - X_{\text{new}} \).

- Mean prediction error: \( E(\bar{X} - X_{\text{new}}) = \)
- Variance of the prediction error: \( V(\bar{X} - X_{\text{new}}) = \)

**Result:** The random variable

\[
\frac{\bar{X} - X_{\text{new}}}{S\sqrt{1 + 1/n}}
\]

has a \( t \) distribution with \( n - 1 \) degrees of freedom.

Using this result, we can write that

\[
P\left[ -t_{\alpha/2,n-1} < \frac{\bar{X} - X_{\text{new}}}{S\sqrt{1 + 1/n}} < t_{\alpha/2,n-1} \right] = 1 - \alpha.
\]

Simplifying, we can write

\[
P\left[ \bar{X} - t_{\alpha/2,n-1}S\sqrt{1 + 1/n} < X_{\text{new}} < \bar{X} + t_{\alpha/2,n-1}S\sqrt{1 + 1/n} \right] = 1 - \alpha
\]
A 100(1 – α)% prediction interval for $X_{new}$ is

$$\bar{X} \pm t_{\alpha/2,n-1}S \sqrt{1 + 1/n},$$

- Valid only if the random sample is from a normal distribution
- Upper and lower prediction bounds can be created as well
  - A lower prediction bound results from replacing $t_{\alpha/2}$ by $t_\alpha$ and discarding the + part
  - A upper prediction bound results from replacing $t_{\alpha/2}$ by $t_\alpha$ and discarding the – part
- Interpretation: The interpretation of a 95% prediction interval is similar to that of a 95% confidence interval: if the prediction interval is calculated for sample after sample, in the long run 95% of these intervals will include the corresponding future values of $X_{new}$.
- Interval width: $2t_{\alpha/2,n-1}S \sqrt{1 + 1/n}$
  - A prediction interval is wider that the corresponding confidence interval.

What happens when sample size $n$ becomes large?
4. Re-visit problem 2 in this handout. We have the following descriptive statistics output.

\[ n = 153, \quad \bar{x} = 135.39, \quad s = 4.59, \quad \text{Minimum} = 122.20, \quad \text{Maximum} = 147.70. \]

- Compute a 95% confidence interval for the true average ultimate tensile strength.

- Now compute a 95% prediction interval for a new observation.

- Compare the prediction interval with the confidence interval.
Confidence Intervals for the Variance and Standard Deviation

So far we have focused on constructing confidence intervals for population mean. However, there are occasions one needs to construct confidence intervals for population variance or standard deviation.

Setup: We have a random sample $X_1, X_2, \ldots, X_n$ from a normal population distribution with unknown mean $\mu$ and unknown variance $\sigma^2$.

Recall: We estimate the population variance $\sigma^2$ by the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$ 

We have shown in previous chapters that $S^2$ is indeed an unbiased estimator for $\sigma^2$.

Result: Suppose $X_1, X_2, \ldots, X_n$ from a normal population distribution mean $\mu$ and variance $\sigma^2$. Then the quantity

$$\frac{n-1}{\sigma^2} S^2$$

has a chi-squared ($\chi^2$) distribution with $n-1$ degrees of freedom.

Properties of $\chi^2$ distribution

- We denote a $\chi^2$ distribution with $\nu$ degrees of freedom as $\chi^2_{\nu}$.
- The density of any $\chi^2$ distribution is defined on positive numbers, that is, random samples from this distribution can not take negative values.
- Each $\chi^2$ has a positive skewness (long right tail).
- As degrees of freedom becomes larger and larger, the $\chi^2$ density curves becomes more and more symmetric.

To compute confidence intervals, we need the critical values of $\chi^2$ distributions:
Confidence intervals for population variance and standard deviation

- A 100(1 − α)% $\chi^2$-confidence interval for the population variance $\sigma^2$ can be formed as

$$\left( \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}} \right)$$

- A 100(1 − α)% $\chi^2$-confidence interval for the population standard deviation $\sigma$ can be formed as

$$\left( \sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}}, \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}} \right)$$

- Note that these interval are not symmetric.

- Valid only if the population distribution is normal.

5. The amount of lateral expansion (mils) was determined for a sample of $n = 9$ pulsed-power gas metal arc welds used in LNG ship containment tanks. The resulting sample standard deviation was $s = 52.81$ mils. Assuming that the population distribution is normal, derive a 95% CI for $\sigma^2$ and for $\sigma$. 