Chapter 8.3: Tests Concerning a Population Proportion

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In this chapter, we will learn about testing hypothesis about population proportion $p$.

- Let $p$ denote the proportion of individuals or objects in a population who possess a specified property (e.g., cars with manual transmissions or smokers who smoke a filter cigarette).

- We denote any object or individual with the specified property as a success. Then $p$ is the population proportion of successes.

- Now we observe a random sample of size $n$. Denote the number of successes in the random sample to be $X$.

- An unbiased estimator of $p$ is given by $\hat{p} = X/n$ (sample proportion).

We know the following results.

- **Result 1:** The number of successes $X$ has a binomial distribution $Bin(n,p)$.

- **Result 2:** When $n$ is large $[np \geq 10$ and $n(1-p) \geq 10]$ then we have

$$\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right),$$

approximately.

We consider two tests:

- Large sample tests based on the normality result (Result 2).

- Small sample test directly using the binomial distribution (Result 1).

Recall that we have learned tests for a population mean $\mu$ using large sample test. There, to test the null hypothesis $H_0 : \mu = \mu_0$, we used $\bar{X}$ as a key quantity, and we used the test statistic

$$Z = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}.$$

Notice that the test statistic has the form

$$Z = \frac{\text{Estimator} - \text{Null value}}{\sqrt{\text{variance of the estimator under } H_0}}.$$

In other words, the large sample test statistic is obtained by standardizing the estimator under the assumption that the null hypothesis $H_0$ is true. Then, under $H_0$, $Z$ follows a standard normal distribution $N(0,1)$. We will use this rule to construct test statistic for $p$. 
Suppose that we want to test the null hypothesis $H_0 : p = p_0$, where $p_0$ is a known value (prior belief about the sample proportion). Recall the large sample result (Result 1): When $n$ is large [$np \geq 10$ and $n(1 - p) \geq 10$] then we have

$$\hat{p} = \frac{X}{n} \sim N\left( p, \frac{p(1-p)}{n} \right),$$

approximately.

In this case, we have

Estimator:

Null value:

Variance of the estimator under $H_0$:

Test statistic:

It can be shown that when $n$ is large and $H_0$ is true, the test statistic above follows a $N(0, 1)$ distribution.

Testing $H_0 : p = p_0$ versus alternative $H_a : p \neq p_0$.

Rejection regions for various alternative hypothesis:

<table>
<thead>
<tr>
<th>Alternative Hypothesis</th>
<th>Rejection Region for Level $\alpha$ Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_a : p \neq p_0$</td>
<td>$z \leq -z_{\alpha/2}$ OR $z \geq z_{\alpha/2}$ (two-tailed)</td>
</tr>
<tr>
<td>$H_a : p &gt; p_0$</td>
<td>$z \geq z_{\alpha}$ (upper-tailed)</td>
</tr>
<tr>
<td>$H_a : p &lt; p_0$</td>
<td>$z \leq -z_{\alpha}$ (lower-tailed)</td>
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</tbody>
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Rule of thumb for a large sample size: $np_0 \geq 10$ and $n(1 - p_0) \geq 10$. 
1. In a highly publicized study, doctors claimed that aspirin seems to help reduce heart attacks rate. Suppose that for a particular age group, 17% of men who do not take aspirin regularly have a heart attack in a 3 year period. Suppose a group of 400 men (all from this age group) took an aspirin tablet three times per week. After 3 years, 56 of them had had heart attacks.

(a) At a 5% level of significance, do you agree with the doctor’s claim? Hint: $z_{0.05} = 1.645$.

(b) What if the significance level is set to 10%? Hint: $z_{0.1} = 1.282$.

(c) Suppose that the true proportion of all aspirin-taking men in this age group who would have a heart attack over the next 3 years is 0.15.

Then in part (a), _______ was made. (Choose one)

In part (b), _______ was made. (Choose one)

Type I Error  Type II Error  Correct decision  Cannot tell
2. The American Community Survey, released this month by the Census Bureau, found that 49.7 percent, or 55.2 million, of the nation’s 111.1 million households in 2005 were made up of married couples. The New York Times of Oct 15, 2006 commented that “Married couples, whose numbers have been declining for decades as a proportion of American households, have finally slipped into a minority”. Statistically, do you agree on this conclusion, that is, do you think the proportion of married household IS significantly less than 50% at a significance level of 0.05?
Calculation of Type II error. Recall that Type II error is defined as NOT rejecting $H_0$ when $H_0$ is in fact false.

- Suppose the null hypothesis is $H_0 : p = p_0$.
- Suppose that $H_0$ is indeed false and the true value of $p$ is in fact $p'$.

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<th>Alternative Hypothesis</th>
<th>Type II Error Probability $\beta(p')$ for a Level $\alpha$ Test</th>
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<tr>
<td>$H_a : p &gt; p_0$</td>
<td>$\Phi\left(\frac{p_0 - p' + z_{\alpha} \sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$</td>
</tr>
<tr>
<td>$H_a : p &lt; p_0$</td>
<td>$1 - \Phi\left(\frac{p_0 - p' - z_{\alpha} \sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$</td>
</tr>
<tr>
<td>$H_a : p \neq p_0$</td>
<td>$\Phi\left(\frac{p_0 - p' + z_{\alpha/2} \sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right) - \Phi\left(\frac{p_0 - p' - z_{\alpha/2} \sqrt{p_0(1-p_0)/n}}{\sqrt{p'(1-p')/n}}\right)$</td>
</tr>
</tbody>
</table>

where $\Phi(z)$ is the standard normal cdf.

The required sample size $n$ for which a level $\alpha$ test also has $\beta(p') = \beta$ is:

- for a one-tailed (upper or lower) test
  \[
n = \left(\frac{z_{\alpha} \sqrt{p_0(1-p_0)} + z_{\beta} \sqrt{p'(1-p')}}{p' - p_0}\right)^2,
\]

- for a two-sided test (an approximate solution)
  \[
n = \left(\frac{z_{\alpha/2} \sqrt{p_0(1-p_0)} + z_{\beta} \sqrt{p'(1-p')}}{p' - p_0}\right)^2.
\]
3. A package-delivery service advertises that at least 90% of all packages brought to its office by 9 A.M. for delivery in the same city are delivered by noon that day. Let $p$ denote the true proportion of such packages that are delivered as advertised and consider the hypotheses

$$H_0 : p = 90\% \quad \text{versus} \quad H_a : p < 90\%.$$

(a) If the truth is that only 80% of the packages are delivered as advertised, how likely is it that a level 0.01 test based on $n = 225$ packages fails to detect such a departure from $H_0$?

(b) What should the sample size be to ensure that $\beta(0.8) = 0.01$?
Small sample tests: When sample size is small, testing procedures are based directly on the binomial result (Result 2), that is $X \sim Bin(n, p)$.

- Null hypothesis $H_0: p = p_0$.
- Suppose we are interested in alternative hypothesis $H_a: p > p_0$.
- Observe that
  - Under $H_0$, $X \sim Bin(n, p_0)$.
  - We will reject $H_0$ in favor of $H_a$ for large values of $X$ (explain why). In other words, we will reject $H_0$ is $X \geq c$ for some cutoff value $c$.
  - Thus need to find a value $c$ such that Type I error of the test is preserved at level $\alpha$. 
4. A plastics manufacturer has developed a new type of plastic trash can and proposes to sell them with an unconditional 6-year warranty. To see whether this is economically feasible, 20 prototype cans are subjected to an accelerated life test to simulate 6 years of use. The proposed warranty will be modified only if the sample data strongly suggests that fewer than 90% of such cans would survive the 6-year period. Let $p$ denote the proportion of all cans that survive the accelerated test. The relevant hypotheses are

$$H_0 : p = 0.9 \text{ versus } H_a : p < 0.9.$$ 

A decision will be based on the test statistic $X$, the number among the 20 that survive. Suppose that the observed number of cans that survived the 6-year period is 14. Perform a test at level $\alpha = 0.05$. 
