Chapter 10.2: Multiple Comparisons in ANOVA

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In last chapter, we learned about Analysis of Variance (ANOVA) and how to perform an F test to compare several means. In this chapter we will learn how to estimate the difference between any pairs of means.
In ANOVA, when the computed value of the $F$ test statistic is not significant, we usually terminate the analysis since we cannot identify any difference among the population means. However, when the null hypothesis is rejected (that is, we detect that there are some differences) then one might want to know which of the means are different from each other. A method for carrying out this further analysis is called a multiple comparisons procedure.

One natural idea is to look at all the pair-wise differences and construct confidence intervals. For example, if there are 4 treatments with means $\mu_1, \ldots, \mu_4$, then one would need to look at $\mu_1 - \mu_2, \mu_1 - \mu_3, \ldots, \mu_3 - \mu_4$ (there are 6 of such pair-wise differences in this case). The following method provides simultaneous confidence intervals for all such possible differences.

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**Tukey’s Procedure (the T Method)**

Tukey’s procedure involves the use of another probability distribution called the Studentized range distribution. The distribution depends on two parameters: a numerator df $m$ and a denominator df $v$. Denote $Q_{\alpha, m, v}$ to be the upper $\alpha$ cut-off value (see Appendix Table A.10).

For a specific confidence level $1 - \alpha$, the **simultaneous intervals** for the differences are given by

$$(\bar{X}_i - \bar{X}_j) \pm Q_{\alpha, I, I(J-1)} \sqrt{\frac{MSE}{J}}$$

for every $i$ and $j$.

Each interval that does not include 0 yields the conclusion that the corresponding values of $\mu_i$ and $\mu_j$ differ significantly from one another. An easy way to detect such differences **without computing the actual interval** is as follows:

- First select $\alpha$ and determine the value of $Q_{\alpha, I, I(J-1)}$ (Table A.10)
- Obtain MSE from the ANOVA table
- Compute $w = Q_{\alpha, I, I(J-1)} \sqrt{\frac{MSE}{J}}$
- List the sample means in **increasing order** and underline those pairs that differ by less than $w$
- Any pair of sample means not underscored by the same line corresponds to a pair of population or treatment means that are judged significantly different.
Example 1: An experiment was carried out to compare five different brands of automobile oil filters with respect to their ability to capture foreign material. Let $\mu_i$ denote the true average amount of material captured by brand $i$ filters ($i = 1, \ldots, 5$) under controlled conditions. A sample of nine filters of each brand was used, resulting in the following sample mean amounts: $\bar{x}_1 = 14.5$, $\bar{x}_2 = 13.8$, $\bar{x}_3 = 13.3$, $\bar{x}_4 = 14.3$, and $\bar{x}_5 = 13.1$. The MSE from ANOVA is $MSE = 0.088$. The test statistic value was $f = 37.84$, which is found significant at $\alpha = 0.05$ level. Find out which means are different.

Hint: From Appendix Table A.10 $Q_{0.05,5,40} = 4.04$. 
**Example 2:** An experiment to compare the spreading rates of five different brands of yellow interior latex paint available in a particular area used 4 gallons \((J = 4)\) of each paint. The sample average spreading rates \((\text{ft}^2/\text{gal})\) for the five brands were \(\bar{x}_1 = 462.0, \bar{x}_2 = 512.8, \bar{x}_3 = 437.5, \bar{x}_4 = 469.3,\) and \(\bar{x}_5 = 532.1.\) The computed value of \(F\) was found to be significant at level \(\alpha = .05.\) With \(\text{MSE} = 272.8,\) use Tukey’s procedure to investigate significant differences in the true average spreading rates between brands.

Hint: From Appendix Table A.10 \(Q_{0.05,5,15} = 4.37.\)