Homework #3 -- due Monday, 04 February 2008 *** turn in just starred (*) questions ***

Exercises at the end of Chapter 2: 1, 2, 3*, 6, 10, 20* and Question 5* (similar to Exercise 2.11) from last year's quiz.

Some results that you might find useful:
Result 2.4. $\mathbf{X}^T\mathbf{X}\mathbf{A} = \mathbf{X}^T\mathbf{X}\mathbf{B}$ iff $\mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}$.
Result 2.7. $\hat{\mathbf{b}}$ solves the normal equations $\mathbf{X}^T\mathbf{X}\mathbf{b} = \mathbf{X}^T\mathbf{y}$ if and only if $\hat{\mathbf{b}}$ solves $\mathbf{X}\mathbf{b} = \mathbf{P}_{\mathbf{X}}\mathbf{y}$
Theorem 2.1. $\mathbf{P}_{\mathbf{X}} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^g\mathbf{X}^T$ is the (unique, symmetric) projection matrix onto $\mathcal{C}(\mathbf{X})$.
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$$\begin{bmatrix} 6 & 1 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{29}\begin{bmatrix} 5 & -1 \\ -1 & 6 \end{bmatrix} \quad \begin{bmatrix} 5 & -2 \\ -2 & 3 \end{bmatrix}^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & 3 \\ 1 & 5 & -2 \\ 3 & -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 11/6 & -3/2 & -17/6 \\ -3/2 & 3/2 & 5/2 \\ -17/6 & 5/2 & 29/6 \end{bmatrix} \quad \begin{bmatrix} 5 & -2 & 2 \\ -2 & 3 & -2 \\ 2 & -2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 0 & -1/3 \\ 0 & 1 & 1 \\ -1/3 & 1 & 11/6 \end{bmatrix}$$

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1) Show that the normal equations are always consistent.

2) Prove that if $\mathbf{A}$ is symmetric, and $\mathbf{G}$ is a generalized inverse of $\mathbf{A}$, then $\mathbf{G}\mathbf{A}\mathbf{G}^T$ is a symmetric generalized inverse of $\mathbf{A}$.

5) A design matrix for an analysis of covariance problem takes the form

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \text{x}$$

a) What is the rank of $\mathbf{X}$?

b) Give the dimension of $\mathcal{C}(\mathbf{X})$ and a nonzero vector in it.

c) Give the dimension of $\mathcal{N}(\mathbf{X})$ and a nonzero vector in it.

d) Give the dimension of $\mathcal{C}(\mathbf{X}^T)$ and a nonzero vector in it.

e) Compute $\mathbf{X}^T\mathbf{X}$ and find a generalized inverse for it. (No need to check $\mathbf{A}\mathbf{G}\mathbf{A} = \mathbf{A}$.)

f) Which of the following vectors could be $\mathbf{P_X y}$ for an appropriate response vector $\mathbf{y}$? (just yes or no)

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

g) Using a right hand side you gave above in (f), find ALL solutions to the equations $\mathbf{Xb} = \mathbf{P_X y}$.

h) Which of the following have the same column space as $\mathbf{X}$? (just yes or no)

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$