Homework #7 -- due Friday, 04 April 2008 *** turn in just starred (*) questions ***

Exercises at the end of Chapter 5: 23, 24, 26, 28, 29*

Exercises at the end of Chapter 6: 4, 8, 11, 24, 25

*)(new question, similar to 6.11)
Consider the balanced one-way ANOVA model with $a = 3$ and $n_i = 4$. Find the power of the F-test for testing that the groups are equal at level 0.05 when $\sigma^2 = 1$ and $\alpha_i = i,\ i = 1,\ 2,\ 3$. Also find the power when the sample size doubles to $n_i = 8$.
(See code in cdfncf.sas to get SAS to compute noncentral F probabilities.)

(Questions from 2007 Final)
3) Consider the usual unbalanced one-way ANOVA model:

$$y_{ij} = \mu + \alpha_i + e_{ij}, \quad \text{for } i = 1, ..., a = 3, \text{ and } j = 1, ..., n_i,$$

with $n_1 = n_2 = 3$ and $n_3 = 2$. Assume, as usual, that e_{ij} are iid $N(0, \sigma^2)$. Below are given the design matrix **X** and observed response vector **y**:

$$\mathbf{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 24 \\ 16 \\ 12 \\ 12 \\ 14 \\ 20 \end{bmatrix} \qquad \mathbf{Xb} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

Note that the rank of **X** is 3 and SSE = 82.

a) Construct the normal equations and find the family of solutions to the normal equations.

b) Since the rank of **X** is 3, construct 3 linearly independent estimable functions.

c) Show that the least squares estimators for the functions you gave in (c) do not depend on the choice of solution to the normal equations.

*4) Now we learn that the experiment in (3) was not properly reported to you, that there was another factor, and that the proper model was a two-way crossed model without interaction:

$$y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \quad \text{for } i = 1, ..., a = 3, \text{ and } j = 1, ..., n = 3,$$

but with one missing cell. Assume, as usual, that e_{ij} are iid $N(0, \sigma^2)$. Below are given the the response vector **y**, new design matrix **X** and parameter vector **b**. Also below is my solution vector to the normal equations, and the generalized inverse I used.

$$\mathbf{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 24 \\ 16 \\ 12 \\ 12 \\ 14 \\ 20 \end{bmatrix} \qquad \mathbf{Xb} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \qquad \hat{\mathbf{b}} = \begin{bmatrix} 0 \\ 20 \\ 16 \\ 20 \\ -4 \\ -2 \\ 0 \end{bmatrix}$$

$$(\mathbf{X}^T\mathbf{X})^g = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 1/3 & 1/2 & -1/2 & -1/2 & 0 \\ 0 & 1/3 & 2/3 & 1/2 & -1/2 & -1/2 & 0 \\ 0 & 1/2 & 1/2 & 5/4 & -3/4 & -3/4 & 0 \\ 0 & -1/2 & -1/2 & -3/4 & 11/12 & 7/12 & 0 \\ 0 & -1/2 & -1/2 & -3/4 & 7/12 & 11/12 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that the rank of **X** is 5 and SSE for this model is 64.

a) Find a basis for $\mathcal{N}(\mathbf{X})$.

b) We're interesting in testing the hypothesis of no block effect H: $\beta_1 = \beta_2 = \beta_3$ vs. A: not all equal. Write this hypothesis as H: $\mathbf{K}^T\mathbf{b} = \mathbf{m}$ by giving **K** and **m**.

c) Test the hypothesis H: $\beta_1 = \beta_2 = \beta_3$ vs. A: not all equal by giving the appropriate test statistic and its distribution under H. You can use either the likelihood ratio test approach, or the $(\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m})^T[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^g\mathbf{K}]^{-1}(\mathbf{K}^T\hat{\mathbf{b}} - \mathbf{m})$ approach.

If we were to construct the restricted normal equations, we should know some things about these equations and a solution.

$$\begin{bmatrix} \mathbf{X}^T\mathbf{X} & \mathbf{K} \\ \mathbf{K}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \mathbf{X}^T\mathbf{y} \\ \mathbf{m} \end{bmatrix}$$

d) Give a possible vector for $\hat{\mathbf{b}}_H$, the first part of the solution vector to the restricted normal equations above.

e) Is $\boldsymbol{\theta}$ equal to zero? (on the Final, I just asked for a yes or no answer)