Homework #3 -- due Monday, 02 February 2009 *** turn in just starred (*) questions ***

Exercises at the end of Chapter 2: 1, 2, 3*, 6, 10, 20* and Question 3* from last year's quiz, one version of which is given below.

Some results that you might find useful:
Lemma 2.1. $\mathcal{N}(\mathbf{X}^T\mathbf{X}) = \mathcal{N}(\mathbf{X})$
Result 2.4. $\mathbf{X}^T\mathbf{X}\mathbf{A} = \mathbf{X}^T\mathbf{X}\mathbf{B}$ iff $\mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}$.
Theorem 2.1. $\mathbf{P_X} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^g\mathbf{X}^T$ is the (unique, symmetric) projection matrix onto $\mathcal{C}(\mathbf{X})$.
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1) Show that the normal equations are always consistent.

2) Prove that if **A** is symmetric, and **G** is a generalized inverse of **A**, then $\mathbf{G}\mathbf{A}\mathbf{G}^T$ is a symmetric generalized inverse of **A**.

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix} \quad \begin{bmatrix} 6 & 2 \\ 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/4 & -1/4 \\ -1/4 & 3/4 \end{bmatrix} \quad \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/4 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 & 8 \\ 4 & 4 & 6 \\ 8 & 6 & 12 \end{bmatrix}^{-1} = \begin{bmatrix} 3/2 & 0 & -1 \\ 0 & 1 & -1/2 \\ -1 & -1/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 6 \\ 0 & 2 & 2 \\ 6 & 2 & 12 \end{bmatrix}^{-1} = \begin{bmatrix} 5/2 & 3/2 & -3/2 \\ 3/2 & 3/2 & -1 \\ -3/2 & -1 & 1 \end{bmatrix}$$
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*3) A design matrix for a particular ANACOVA problem takes the form

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

a) What is the rank of **X**?

b) Give the dimension of the following vector spaces:

i) dim $\mathcal{C}(\mathbf{X})$ = ___ ii) dim $\mathcal{N}(\mathbf{X})$ = ___ iii) dim $\mathcal{N}(\mathbf{X}^T)$ = ___

c) Put each of the following five vectors in one of these four vector spaces: $\mathcal{C}(\mathbf{X})$, $\mathcal{N}(\mathbf{X})$, $\mathcal{N}(\mathbf{X}^T)$, $\mathcal{C}(\mathbf{X}^T)$:

$$\begin{bmatrix} 5 \\ 2 \\ 3 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 4 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 8 \\ 6 \\ 2 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

d) Compute $\mathbf{X}^T\mathbf{X}$ and find a generalized inverse for it. (No need to check $\mathbf{AGA} = \mathbf{A}$.)

e) Give a vector from the list in (c) which could be a right hand side ($\mathbf{X}^T\mathbf{y}$) of the normal equations.

f) Using the right hand side you gave above in (e), find ALL solutions to the normal equations $\mathbf{X}^T\mathbf{X}\mathbf{b} = \mathbf{X}^T\mathbf{y}$.

g) Fill in the missing column in the design matrix $\mathbf{W}$ below so that $\mathbf{P_X} = \mathbf{P_W}$.

$$\mathbf{W} = \begin{bmatrix} -1 & 0 & \\ -1 & 0 & \\ -1 & 0 & \\ -1 & 0 & \\ 1 & 2 & \\ 1 & 2 & \end{bmatrix}$$