3) Consider the usual unbalanced one-way ANOVA model:
\[ y_{ij} = \mu + \alpha_i + \epsilon_{ij}, \quad \text{for } i = 1, ..., a = 3, \text{ and } j = 1, ..., n_i, \]
with \( n_1 = n_2 = 3 \) and \( n_3 = 2 \). Assume, as usual, that \( \epsilon_{ij} \) are iid \( \mathcal{N}(0, \sigma^2) \). Below are given the design matrix \( X \) and observed response vector \( y \):

\[
\begin{bmatrix}
y_{11} \\
y_{12} \\
y_{13} \\
y_{21} \\
y_{22} \\
y_{23} \\
y_{31} \\
y_{32}
\end{bmatrix} =
\begin{bmatrix}
16 \\
14 \\
15 \\
10 \\
14 \\
12 \\
14 \\
18
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8
\end{bmatrix}
\]

Note that the rank of \( X \) is 3 and \( \text{SSE} = 18 \).

a) Construct the normal equations and find the family of solutions to the normal equations.

b) Since the rank of \( X \) is 3, construct 3 linearly independent estimable functions.

c) Show that the least squares estimators for the functions you gave in (c) do not depend on the choice of solution to the normal equations.
*4) Now we learn that the experiment in (3) was not properly reported to you, that there was another factor, and that the proper model was a two-way crossed model without interaction:

\[ y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}, \quad \text{for } i = 1, ..., a = 3, \text{ and } j = 1, ..., n = 3, \]

but with one missing cell. Assume, as usual, that \( e_{ij} \) are iid \( N(0, \sigma^2) \). Below are given the the response vector \( y \), new design matrix \( X \) and parameter vector \( b \). Also below is my solution vector to the normal equations, and the generalized inverse \( I \) I used.

\[
y = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 16 \\ 14 \\ 15 \\ 10 \\ 14 \\ 12 \\ 14 \\ 18 \end{bmatrix} \quad Xb = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ 12 \\ 16 \\ -1 \\ 1 \\ 0 \end{bmatrix}
\]

Note that the rank of \( X \) is 5 and SSE for this model is 12.

a) Find a basis for \( \mathcal{N}(X) \).

b) We're interesting in testing the hypothesis of no block effect \( H: \beta_1 = \beta_2 = \beta_3 \) vs. \( A: \) not all equal. Write this hypothesis as \( H: \mathbf{K}^Tb = \mathbf{m} \) by giving \( \mathbf{K} \) and \( \mathbf{m} \).

c) Test the hypothesis \( H: \beta_1 = \beta_2 = \beta_3 \) vs. \( A: \) not all equal by giving the appropriate test statistic and its distribution under \( H \). You can use either the likelihood ratio test approach, or the \( (\mathbf{K}^T\hat{b} - \mathbf{m})^T[\mathbf{K}^T(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{K}]^{-1}(\mathbf{K}^T\hat{b} - \mathbf{m}) \) approach.

If we were to construct the restricted normal equations, we should know some things about these equations and a solution.

\[
\begin{bmatrix} X^T X & K \\ K^T & 0 \end{bmatrix} \begin{bmatrix} b \\ \theta \end{bmatrix} = \begin{bmatrix} X^T y \\ m \end{bmatrix}
\]

d) Give a possible vector for \( \hat{b}_H \), the first part of the solution vector to the restricted normal equations above.

e) Is \( \theta \) equal to zero? (on the Final, I just asked for a yes or no answer)