1. Let \( f: E^n \rightarrow E^1 \) be a differentiable function. Prove that the gradient vector \( \nabla f(x) \) is the vector of partial derivatives \( \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)^T \). [Hint: consider the definition of partial derivative: the partial derivative of \( f(x_1, x_2, \ldots, x_n) \) with respect to \( x_i \), written \( \frac{\partial f}{\partial x_i} \), is]

\[
\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}.
\]

2. Obtain expressions for \( \nabla f(x) \) and \( \nabla^2 f(x) \) for the following functions \( f: E^n \rightarrow E^1 \):
   (i) \( f(x) = a^T x \), where \( a \in E^n \) is a given constant vector;
   (ii) \( \frac{1}{2} x^T A x + b^T x \), \( A \in \mathbb{R}^{n \times n} \), symmetric, and constant; \( b \in E^n \) constant.
   (iii) \( x^T A x \), where \( A \) is \( n \times n \) but not necessarily symmetric.

3. Obtain expressions for the gradient \( \nabla f(x) \) and Hessian \( \nabla^2 f(x) \) of the function
   \[ f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2. \]
   This function is called Rosenbrock's function or the "banana" function because of the shape of its contours; we will discuss it periodically throughout the course.

4. For what values of \( x \) is the standard normal probability distribution (density function)
   \[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]
   convex? concave?

5. Determine the Hessian matrix of the function \( f(x) = x_1^2 + x_2^2 + 2x_3^2 - x_1x_2 - x_2x_3 - x_3x_1 \) and find the eigenvectors and eigenvalues of \( \nabla^2 f(x) \). Is \( f(x) \) convex, concave, or neither?

6. Maximize \( f(x) = xe^{-x^2} \).

7. Find the critical points of \( f(x) = x^4 + 4x^3 + 6x^2 + 4x \); categorize the critical points as local minimum points, local maximum points, or neither.

8. Maximize \( f(x_1, x_2, x_3) = x_1(x_2 - 1) + x_3(x_3^2 - 3) \)