1. Let \( f: \mathbb{E}^n \rightarrow \mathbb{E}^1 \) be a differentiable function. Show that the gradient vector \( \nabla f(x) \) is the vector of partial derivatives 
\[
\left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)^T.
\]
(Hint: In the definition of differentiability, choose \( x = \mathbf{x} + h^i \) where \( h^i \in \mathbb{E}^n \) has value \( h \in \mathbb{E}^1 \) for component \( i \) and value 0 for all other components. Also recall the definition of partial derivative:

the partial derivative of \( f(x_1, x_2, \ldots, x_n) \) with respect to \( x_i \), written \( \frac{\partial f}{\partial x_i} \), is
\[
\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, \ldots, x_i + \Delta x_i, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_i}.
\]

2. Obtain expressions for \( \nabla f(x) \) and \( \nabla^2 f(x) \) for the following functions \( f: \mathbb{E}^n \rightarrow \mathbb{E}^1 \):

(i) \( f(x) = a^T x \), where \( a \in \mathbb{E}^n \) is a given constant vector;

(ii) \( \frac{1}{2} x^T A x \), where \( A \) is an \( n \times n \) matrix

(iii) \( \frac{1}{2} x^T A x + b^T x \), where \( A \) is an \( n \times n \) symmetric matrix; \( b \in \mathbb{E}^n \).

3. Obtain expressions for the gradient \( \nabla f(x) \) and Hessian \( \nabla^2 f(x) \) of the function
\[
f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.
\]
(This function is called Rosenbrock’s function (or the “banana” function because of the shape of its contours); we will discuss it periodically throughout the course).

4. Consider the following unconstrained problem:
\[
\text{minimize } f(x_1, x_2) = x_1^2 - x_1 x_2 + 2x_2^2 - 2x_1 + e^{x_1 + x_2}
\]

a. Write the first-order necessary optimality conditions.

b. Is \( \mathbf{x} = (0, 0)^T \) an optimal solution? If not, identify a direction \( \mathbf{d} \in \mathbb{E}^2 \) along which the function would decrease.

c. Minimize the function \( f \) starting from \( (0, 0)^T \) along the direction \( \mathbf{d} \) obtained in part b.

5. Consider the problem to minimize \( f : \mathbb{E}^n \rightarrow \mathbb{E}^1 \) where
\[
f(x) = \| Ax - b \|^2 = (Ax - b)^T (Ax - b)
\]
where \( A \) is an \( m \times n \) matrix and \( b \) is an \( m \times 1 \) vector.

a. Give a geometric interpretation of the problem.

b. Write a necessary condition for optimality.

c. Is the optimal solution unique? Why or why not?

d. Can you give a closed form solution of the optimal solution point \( \mathbf{x} \).

e. Solve the problem for \( A \) and \( b \) given below:
\[
A = \begin{bmatrix}
1 & -1 & 0 \\
0 & 2 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad b = \begin{bmatrix}
2 \\
1 \\
1 \\
0
\end{bmatrix}
\]

Use the following theorem to do exercise #6:

If \( f: \mathbb{E}^n \rightarrow \mathbb{E}^1 \) is continuous on a closed and bounded region, then \( f \) has a global maximum and global minimum on that region.

6. Find the global minimum and global maximum of \( f(x_1, x_2) = x_1^2 - x_2^2 + 3 \) over \(-1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\).
Use the following definition and theorem to do exercises #7 and #8 (you don’t have to prove the theorem):

**Def.** A continuous function \( f(x) \) defined on all of \( E^n \) is called **coercive** if

\[
\lim_{\|x\| \to \infty} f(x) = +\infty
\]

**Thm.** Let \( f(x) \) be a continuous function defined on \( E^n \). If \( f(x) \) is coercive, then \( f(x) \) has at least one global minimizing point. If, in addition, the first partial derivatives of \( f(x) \) exist on all of \( E^n \), then these global minimizers can be found among the critical points of \( f(x) \).

7. Let \( f(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 \).
   i) show that for each fixed \( \overline{x}_2 \), we have \( \lim_{|x_1| \to \infty} f(x_1, \overline{x}_2) = \infty \);
   ii) show that for each fixed \( \overline{x}_1 \), we have \( \lim_{|x_2| \to \infty} f(\overline{x}_1, x_2) = \infty \)
   iii) but \( f(x_1, x_2) \) is **not** coercive.

8. Let \( f(x_1, x_2) = x_1^4 - 4x_1x_2 + x_2^4 \).
   i) Show that \( f \) is coercive \([f \text{ can be expressed as } f(x_1, x_2) = (x_1^2 + x_2^2)(1 - \frac{4x_1x_2}{x_1^2 + x_2^2})]\);
   ii) show that \( \nabla^2 f(x_1, x_2) \) is not positive definite on \( E^2 \) \([\text{evaluate } \nabla^2 f(x_1, x_2) \text{ at } (\frac{1}{2}, \frac{1}{2})^T]\);
   iii) use the above theorem to show that \((-1, -1)^T\) and \((1, 1)^T\) are both global minimizers of \( f(x_1, x_2) \).