1. If \( A \subseteq E^n \) is a subspace of \( E^n \), show that \( A' = A^\perp \), where \( A' \) is the dual cone of \( A \) and \( A^\perp \) is the orthocomplement of \( A \).

**Notes:**
1. \( A \subseteq E^n \) is a subspace of \( E^n \) if for any \( x, y \in A \) and any scalars \( a, b \), \( ax + by \in A \).
2. If \( A \subseteq E^n \) is a subspace of \( E^n \), the orthocomplement \( A^\perp \equiv \{ x \in E^n : x^T y = 0 \ \text{for all}\ y \in A \} \).

2. If \( A_1 \) and \( A_2 \) are nonempty subsets of \( E^n \) with \( A_1 \subseteq A_2 \), show that \( A_2^\prime \subseteq A_1^\prime \).

3. If \( X = \{(x_1, x_2) \in E^2 : x_2 \geq -x_1^2 \} \), find \( S(X, \overline{x}) \) where \( \overline{x} = (0, 0) \).

4. Let \( X \) be a subset of \( E^n \), and let \( \overline{x} \in \text{int} \ X \). Show that the cone of tangents \( S(X, \overline{x}) \) of \( X \) at \( \overline{x} \) is \( E^n \). (A point \( x \) is in the interior of \( X \), denoted \( \text{int} \ X \), if \( N_r(x) \subseteq X \) for some \( \epsilon > 0 \), where \( N_r(x) = \{ y : \| y - x \| < \epsilon \} \) is an \( \epsilon \)-neighborhood around the point \( x \).

5. Consider the minimization problem 
   
   \[
   \min f(x), \quad x \in X
   \]
   
   where \( X \subseteq E^n \) and \( f : E^n \rightarrow E^1 \) is differentiable.

   We showed in class that a necessary condition for \( \overline{x} \) to be a minimum of \( f \) is \( \nabla f(\overline{x}) \in (S(X, \overline{x}))^\prime \). In the special case where \( X = E^n \), what is the form of the above necessary condition? (Recall that if \( K \) is a cone, then \( K^\prime \equiv \{ y \in E^n : y^T x \geq 0 \ \text{for all}\ x \in K \} \) is the dual cone of \( K \).

6. Let \( f : E^n \rightarrow E^1 \) be a convex function. Show that \( \xi \) is a subgradient of \( f \) at \( \overline{x} \) if and only if the hyperplane \( \{ (\overline{x}, y) : y = f(\overline{x}) + \xi^T (\overline{x} - \overline{x}) \} \) supports \( \text{epi} f \) at \( [\overline{x}, f(\overline{x})] \).

**Note:**
Let \( X \) be a nonempty subset of \( E^n \) and let \( \overline{x} \in \partial X \), where \( \partial X \) denotes the boundary of \( X \). A hyperplane \( H = \{ x : p^T (x - \overline{x}) = 0 \} \) is called a supporting hyperplane of \( X \) at \( \overline{x} \) if either \( X \subseteq H^+ = \{ x : p^T (x - \overline{x}) \geq 0 \} \) or else \( X \subseteq H^- = \{ x : p^T (x - \overline{x}) \leq 0 \} \). (Definition: \( \overline{x} \in \partial X \) if \( N_r(\overline{x}) \) contains at least one point in \( X \) and one point not in \( X \) for every \( \epsilon > 0 \).

**Interesting facts relating optimization and \( \pi \):**

(no work is required for this problem; just read and expand your mathematical horizons).

Everyone knows that \( \pi = 3.14159... \) is the ratio of the circumference of a circle to its diameter, given that \( E^2 \) is equipped with the usual Euclidean metric. Giving \( E^2 \) other metrics, however, alters the value of \( \pi \).

For example, under the taxicab metric \( d_1 \) on the plane, defined by

\[
d_1((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|
\]

circles are diamond-shaped and the value of \( \pi \) is 4. For the max-norm metric \( d_\infty \) given by

\[
d_\infty((x_1, y_1), (x_2, y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|)
\]

circles are square and again \( \pi \) is 4.

The above metrics are special cases of a class of metrics

\[
d_p((x_1, y_1), (x_2, y_2)) = \sqrt[p]{|x_2 - x_1|^p + |y_2 - y_1|^p},
\]
defined only for \( p \geq 1 \). The table below gives approximate values of \( \pi \) for various values of \( p \).
The table suggests, and it is indeed true, that the minimum value of $\pi_p$ for this class of metrics is $\pi_2 = \pi = 3.14159...$