Martingales Again

• As in discrete time, we need a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\): \(\mathcal{F}_t\) is a sub-\(\sigma\)-field of \(\mathcal{F}\) and for \(0 \leq s < t\), \(\mathcal{F}_s \subseteq \mathcal{F}_t\).

• The natural filtration generated by a stochastic process \(\{X_t\}_{t \geq 0}\) is \(\{\mathcal{F}_t^X\}_{t \geq 0}\);
  – Here \(\mathcal{F}_t^X\) is the smallest \(\sigma\)-field with respect to which all \(X_s\) are measurable, \(0 \leq s \leq t\).
• \( \{M_t\}_{t \geq 0} \) is a \( (\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \)-martingale if

\[
\mathbb{E}^\mathbb{P}[|M_t|] < \infty \text{ for all } t \geq 0,
\]

and for any \( 0 \leq s \leq t \),

\[
\mathbb{E}^\mathbb{P}[M_t|\mathcal{F}_s] = M_s.
\]

• More generally, \( \{M_t\}_{t \geq 0} \) is a \textit{local} \( (\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \)-martingale if there exists a sequence of stopping times \( \{T_n\}_{n \geq 1} \) such that:

- \( \{M_{t \wedge T_n}\}_{t \geq 0} \) is a \( (\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \)-martingale for each \( n \), and

- \( \mathbb{P}[T_n \to \infty \text{ as } n \to \infty] = 1. \)
Suppose that \( \{W_t\}_{t \geq 0} \) is standard Brownian motion, and \( \{\mathcal{F}_t\}_{t \geq 0} \) is the natural filtration. Then:

- \( \{W_t\}_{t \geq 0} \) is a \((\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\)-martingale;
- \( \{W_t^2 - t\}_{t \geq 0} \) is a \((\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\)-martingale;
- \( \left\{ \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right) \right\}_{t \geq 0} \) is a \((\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})\)-martingale.
• Optional Stopping Theorem: if \( \{M_t\}_{t \geq 0} \) is a continuous \( (\mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}) \)-martingale (or more generally, almost surely càdlàg), and if \( \tau_1 \leq \tau_2 \) are two bounded stopping times, then

\[
\mathbb{E}[|M_{\tau_2}|] < \infty,
\]

and

\[
\mathbb{E}[M_{\tau_2} | \mathcal{F}_{\tau_1}] = M_{\tau_1},
\]

with \( \mathbb{P} \)-probability 1.
• We can use this theorem for instance to find the moment generating function (Laplace transform) of the distribution of the hitting time $T_a$,

$$E\left[e^{-\theta T_a}\right].$$

• Suppose that $a > 0$; recall that

$$M_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t\right)$$

is a martingale.

• Take $\tau_1 = 0$ and $\tau_2 = T_a$. 
• So

\[ 1 = M_0 \overset{?}{=} \mathbb{E}[M_{T_a}] = e^{\sigma \alpha} \mathbb{E}\left[e^{-\frac{\sigma^2}{2}T_a}\right]. \]

• The argument fails at "\(=\)”, because \(T_a\) is not bounded.

• It can be fixed by taking \(\tau_2 = T_a \wedge n\), and letting \(n \to \infty\).

• Correct:

\[
1 = M_0 = \mathbb{E}[M_{T_a \wedge n}]
= \mathbb{E}[M_{T_a \wedge n}1\{T_a \leq n\}] + \mathbb{E}[M_{T_a \wedge n}1\{T_a > n\}]
= \mathbb{E}[M_{T_a} 1\{T_a \leq n\}] + \mathbb{E}[M_n 1\{T_a > n\}] \to \mathbb{E}[M_{T_a}] + 0.
\]
• So

\[ E \left[ e^{-\sigma^2 T_a} \right] = e^{-\sigma a}, \]

or

\[ E \left[ e^{-\theta T_a} \right] = e^{-\sqrt{2}\theta a}. \]

• For \( a < 0 \), we use

\[ M_t = \exp \left( -\sigma W_t - \frac{\sigma^2}{2} t \right) \]

as the martingale, and find the general result

\[ E \left[ e^{-\theta T_a} \right] = e^{-\sqrt{2}\theta |a|}. \]