Pricing a Collateralized Debt Obligation

- A Collateralized Debt Obligation (CDO) is a structured financial transaction; it is one kind of Asset-Backed Security (ABS).

- A Manager designs a portfolio of debt obligations, such as corporate bonds (in a CBO) or commercial loans (in a CLO).

- The Manager recruits a number of investors who buy rights to parts of the portfolio.
• Each investor buys the right to receive certain cash flows derived from the portfolio, divided into *tranches*.

• The *senior tranche* has first call on the cash flows (interest and principal), up to a set percentage.

• The *junior tranche* has next call, again up to a set percentage.

• Remaining cash flows are passed through to the *equity tranche*. 
• Simplified example: suppose that the portfolio consists of two corporate bonds, $B_1$ and $B_2$, and neither pays interest.

• The bonds are priced at $b_1$ and $b_2$ at $t = 0$.

• Each bond returns 1 at $t = T$ if its issuer is not in default, and 0 if the issuer has defaulted.

• The matrix $D$ is:

<table>
<thead>
<tr>
<th>In default:</th>
<th>Neither</th>
<th>Issuer 1</th>
<th>Issuer 2</th>
<th>Both</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash</td>
<td>$e^{rT}$</td>
<td>$e^{rT}$</td>
<td>$e^{rT}$</td>
<td>$e^{rT}$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$B_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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</tbody>
</table>
• \( D \) is \( 3 \times 4 \), so \( N = 3 < n = 4 \), and the market is not complete.

• Now

\[
S_0 = \begin{pmatrix}
1 \\
b_1 \\
b_2
\end{pmatrix}.
\]

• If \( 0 \leq b_i < e^{-rT}, i = 1, 2 \), then \( S_0 = D\psi \) where

\[
\psi = e^{-rT} \begin{pmatrix}
(1 - p_1)(1 - p_2) \\
p_1(1 - p_2) \\
(1 - p_1)p_2 \\
p_1p_2
\end{pmatrix}
\]

and \( p_i = 1 - e^{rT}b_i, i = 1, 2 \).
Clearly $\psi$ is a state price vector, and the corresponding risk-neutral measure $\mathbb{Q}$ implies that the probabilities of default are $p_1$ and $p_2$.

It also implies independence of the events of default.

But because the market is not complete, other state price vectors and other risk-neutral measures exist.
Suppose that the CDO has just a senior tranche $S$ and the equity tranche $E$, and each receives 50\% of the cash flows (which are only the return of principal at $t = T$).

With these added to the market, $D$ becomes

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• Actually, one of $S$ and $E$ is redundant, because $S + E = B_1 + B_2$. For convenience, we drop $E$.

• If the senior tranche has price $s$ at $t = 0$, we find

$$\psi = D^{-1}S_0 = \begin{pmatrix} b_1 + b_2 - s \\ s - b_1 \\ s - b_2 \\ e^{-rT} - s \end{pmatrix}.$$ 

• For the market to be arbitrage-free, we must have

$$\max(b_1, b_2) < s < \min(e^{-rT}, b_1 + b_2).$$

• Note that seniority means that $S$ costs more than either bond.
• If $Q_s[.]$ is the corresponding risk-neutral measure, we find that the probability of default of issuer 1 is

$$e^{rT}(s - b_1 + e^{-rT} - s) = 1 - e^{rT}b_1 = p_1,$$

as before, and similarly $p_2$ for issuer 2.

• But the probability that they both default is $1 - se^{rT}$, and this equals $p_1p_2$ only when

$$s = b_1 + b_2 - e^{rT}b_1b_2.$$

• Typically, $s < b_1 + b_2 - e^{rT}b_1b_2$, which means that the probability of both issuers defaulting is higher than it would be under independence.
• In this case, there is *positive dependence* between the events of default.

• Dependence is often modeled using a *Gaussian copula*.

• Suppose that default is associated with random variables $Z_1$ and $Z_2$, normally distributed with mean 0 and variance 1, and correlation $\rho$.

• Issuer 1 defaults if and only if $Z_1 < \Phi^{-1}(p_1)$, and similarly issuer 2.
• The probability that both default is a function of $\rho$.

• If $\rho = 0$, events of default are independent.

• The value of $\rho$ that corresponds to the risk-neutral probability of both issuers defaulting is the *implied correlation*. 